An unbiased Nitsche's approximation of the frictional contact between two elastic structures

Franz Chouly, Rabii Mlika, Yves Renard

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Abstract

Most of the numerical methods dedicated to the contact problem involving two elastic bodies are based on the master/slave paradigm. It results in important detection difficulties in the case of self-contact and multi-body contact, where it may be impractical, if not impossible, to a priori nominate a master surface and a slave one. In this work we introduce an unbiased finite element method for the finite element approximation of frictional contact between two elastic bodies in the small deformation framework. In the proposed method the two bodies expected to come into contact are treated in the same way (no master and slave surfaces). The key ingredient is a Nitsche-based formulation of contact conditions, as in [7]. We carry out the numerical analysis of the method, and prove its well-posedness and optimal convergence in the H^1-norm. Numerical experiments are performed to illustrate the theoretical results and the performance of the method.

Keywords: Two deformable bodies contact problem, Nitsche's method, Unbiased /Master-Slave formulation, Tresca friction, Finite element method.

Introduction

Although being a very rich subject in the past, contact computational mechanics for deformable bodies in small or large strain is still the subject of intensive research. The most common approach to treat the problem of two deformable bodies in contact is known as the master/slave formulation. In this approach one distinguishes between a master surface and a slave one on which one prescribes the non penetration condition. A breakdown of this formulation and the contact problem can be found in Laursen's work [17, 18](see also [19]) and a presentation of discretization schemes and numerical algorithms for mechanical contact is given in [24]. This approach is confronted with important difficulties especially in the case of self-contact and multi-body contact where it is impossible or impractical to a priori nominate a master surface and a slave one. Automating the detection and the separation between slave and master surfaces in these cases may generate a lack of robustness since it may create detection problem.

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If the master/slave formulation consists in a natural extension of the contact treatment between a deformable body and a rigid ground, it has no complete theoretical justification. Consequently, to avoid these difficulties, we give in this article an unbiased formulation of the two elastic bodies contact problem in the small strain framework. In this formulation we do not distinguish between a master surface and a slave one since we impose the non penetration and the friction conditions on both of them. Unbiased contact and friction formulation have been considered before in [22] and references therein. In there, the authors present a numerical study of the method and make as of a penalized formulation of contact and friction. The terms two-pass and two-half-pass are also used in literature to describe this type of methods.

In the present study, we use Nitsche's method for contact and friction (see [5,7]). Nitsche's method is a promising method to treat frictionless unilateral contact in small strain assumption. It is an extension of the method proposed in 1971 by J. Nitsche to treat Dirichlet condition within the variational formulation without adding Lagrange multipliers [20]. Nitsche's method has been widely applied on problems involving linear conditions on the boundary of a domain or in the interface between sub-domains: see, e.g. [23] for the Dirichlet problem or [1] for domain decomposition with non matching meshes. More recently, in [12] and [14] it has been adapted for bilateral (persistent) contact, which still involves linear boundary conditions on the contact zone. A Nitsche-based formulation for the Finite Element discretization of the unilateral (non-linear) contact problem in linear elasticity was introduced in [5] and generalized in [7] to encompass symmetric and non-symmetric variants. A simple adaptation to Tresca's friction of the Nitsche-based Finite Element Method is proposed in [4].
Conversely to standard penalization techniques (see [6,16]), the resulting method is consistent. Moreover, unlike mixed methods (see [13,15]), no additional unknown (Lagrange multiplier) is needed. Thus, the adaptation of the method to an unbiased contact description is quite easier. In fact, since Nitsche's method uses the contact stress as a multiplier, it is very simple to divide this contact effort equitably on both of contact surfaces.
This study can be seen as a first step in the construction of a method taking into account contact between two elastic solids and self-contact in large transformations in the same formalism. The present formulation, in small deformations, allows us to ensure the consistency, the convergence and the optimality of the method. In this context, the aim of this paper is to provide an unbiased description of the contact and Tresca friction conditions and to use Nitsche's method to give a variational formulation of the problem. The formulation uses an additional parameter θ as in [7], allowing us to introduce some interesting variants acting on the symmetry / skew-symmetry / non-symmetry of the discrete formulation. Moreover, a unified analysis of all these variants can be performed. We provide, as well, theoretical and numerical verifications of the proposed method. Some mathematical analysis need to be performed to prove the consistency of the method, its well-posedness and its optimal convergence.

In section 1 we build an unbiased formulation of the two elastic bodies frictional (Tresca) contact problem. This formulation will be based on Nitsche's method. To prove the efficiency of the method (20), we carry out some mathematical analysis on it in section 2. In the last section 3 of this paper, we present the results of several two/three-dimensional numerical tests. The tests cover a study of convergence in H^1-norm of the global relative error and in L^2-norm for the contact pressure error with different values of the generalization parameter θ and the Nitsche's parameter γ_0. The open source environment GetFEM++[1] is used to perform the tests.

[1] http://download.gna.org/getfem/html/homepage/index.html

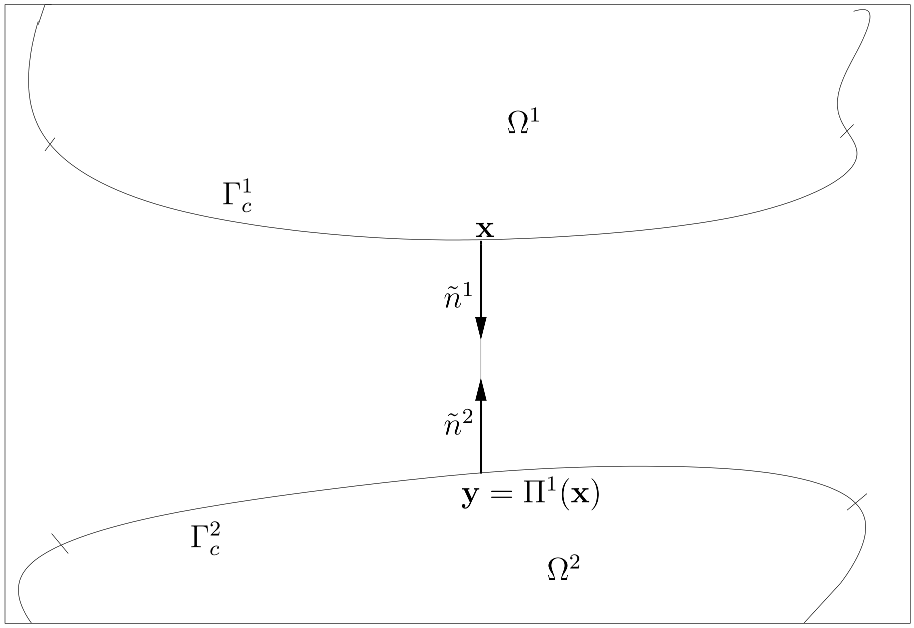


Figure 1: Example of definition of $\tilde{\mathbf{n}}^i$

1 Setting of the problem

1.1 Formal statement of the two bodies contact problem

We consider two elastic bodies (1) and (2) expected to come into contact. To simplify notations, a general index (i) is used to represent indifferently the body (1) or (2). Let Ω^i be the domain in $\mathbb{R}^d$ occupied by the reference configuration of the body (i), with $d = 2$ or 3. Small strain assumption is made, as well as plane strain when $d = 2$. We suppose that the boundary $\partial\Omega^i$ of each body consists in three non-overlapping parts Γ_D^i , Γ_N^i and Γ_C^i. On Γ_D^i (resp Γ_N^i) displacements $\mathbf{u}^i$ (resp. tractions $\mathbf{t}^i$) are given. The body is clamped on Γ_D^i for the sake of simplicity. In addition each body can be subjected to a volumic force $\mathbf{f}^i$ (such as gravity). We denote by Γ_C^i a portion of the boundary of the body (i) which is a candidate contact surface with an outward unit normal vector $\mathbf{n}^i$. The actual surface on which a body comes into contact with the other one is not known in advance, but is contained in the portion Γ_C^i of $\partial\Omega^i$.
Furthermore let us suppose that Γ_C^i is smooth. For the contact surfaces, let us assume a sufficiently smooth one to one application (projection for instance) mapping each point of the first contact surface to a point of the second one:

$$\Pi^1 : \Gamma_C^1 \to \Gamma_C^2.$$

Let J^1 be the Jacobian of the transformation Π^1 and $J^2 = \dfrac{1}{J^1}$ the Jacobian of $\Pi^2 = (\Pi^1)^{-1}$. We suppose in the following that $J^1 > 0$.

We define on each contact surface a normal vector $\tilde{\mathbf{n}}^i$ (see Figure1) such that:

$$\tilde{\mathbf{n}}^i(\mathbf{x}) = \begin{cases} \dfrac{\Pi^i(\mathbf{x}) - \mathbf{x}}{\|\Pi^i(\mathbf{x}) - \mathbf{x}\|} & \text{if } \mathbf{x} \neq \Pi^i(\mathbf{x}), \\ \mathbf{n}^i & \text{if } \mathbf{x} = \Pi^i(\mathbf{x}). \end{cases}$$

Note that $\tilde{\mathbf{n}}^1 = -\tilde{\mathbf{n}}^2 \circ \Pi^1$ and $\tilde{\mathbf{n}}^2 = -\tilde{\mathbf{n}}^1 \circ \Pi^2$.

The displacements of the bodies, relatively to the fixed spatial frame are represented by $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$, where $\mathbf{u}^i$ is the displacement field of body (i).
The contact problem in linear elasticity consists in finding the displacement field $\mathbf{u}$ verifying the equations (1) and the contact conditions described hereafter:

$$\mathbf{div}\,\boldsymbol{\sigma}^i(\mathbf{u}^i) + \mathbf{f}^i = \mathbf{0} \quad \text{in } \Omega^i, \tag{1a}$$
$$\boldsymbol{\sigma}^i(\mathbf{u}^i) = A^i \boldsymbol{\varepsilon}(\mathbf{u}^i) \quad \text{in } \Omega^i, \tag{1b}$$
$$\mathbf{u}^i = \mathbf{0} \quad \text{on } \Gamma_D^i, \tag{1c}$$
$$\boldsymbol{\sigma}^i(\mathbf{u}^i)\mathbf{n}^i = \mathbf{t}^i \quad \text{on } \Gamma_N^i, \tag{1d}$$

where $\boldsymbol{\sigma}^i = \sigma^i_{(j,k)}, 1 \leq j, k \leq d$, stands for the stress tensor field and $\mathbf{div}$ denotes the divergence operator of tensor valued functions. The notation $\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)$ represents the linearized strain tensor field and A^i is the fourth order symmetric elasticity tensor on Ω^i having the usual uniform ellipticity and boundedness property.
For any displacement field $\mathbf{v}^i$ and for any density of surface forces $\boldsymbol{\sigma}^i(\mathbf{v}^i)\mathbf{n}^i$ defined on $\partial\Omega_i$ we adopt the following notation:

$$\mathbf{v}^i = v_n^i \tilde{\mathbf{n}}^i + \mathbf{v}_t^i \text{ and } \boldsymbol{\sigma}^i(\mathbf{v}^i)\mathbf{n}^i = \sigma_n^i(\mathbf{v}^i)\tilde{\mathbf{n}}^i + \boldsymbol{\sigma}_t^i(\mathbf{v}^i),$$

where $\mathbf{v}_t^i$ (resp $\boldsymbol{\sigma}_t^i(\mathbf{v}^i)$) are the tangential components of $\mathbf{v}^i$ (resp $\boldsymbol{\sigma}^i(\mathbf{v}^i)\mathbf{n}^i$).
We define an initial normal gap representing the normal distance between a point $\mathbf{x}$ of Γ_C^i and its image on the other body: $g_n^i = (\Pi^i(\mathbf{x}) - \mathbf{x}) \cdot \tilde{\mathbf{n}}^i$.
We define, as well, the relative normal displacements $[\![u]\!]_n^1 = (\mathbf{u}^1 - \mathbf{u}^2 \circ \Pi^1) \cdot \tilde{\mathbf{n}}^1$ and $[\![u]\!]_n^2 = (\mathbf{u}^2 - \mathbf{u}^1 \circ \Pi^2) \cdot \tilde{\mathbf{n}}^2$.

Remark 1.1. *Note that:* $g_n^1 \circ \Pi^2 = g_n^2$ *and* $g_n^2 \circ \Pi^1 = g_n^1$; $[\![u]\!]_n^1 \circ \Pi^2 = [\![u]\!]_n^2$ *and* $[\![u]\!]_n^2 \circ \Pi^1 = [\![u]\!]_n^1$.

In order to obtain an unbiased formulation of the contact problem we prescribe the contact conditions deduced form the Signorini problem conditions (see [16]) on the two surfaces in a symmetric way. Thus, the conditions describing contact on Γ_C^1 and Γ_C^2 are:

$$\left.\begin{aligned} [\![u]\!]_n^1 &\leq g_n^1 && \text{(2a)}\\ \sigma_n^1(\mathbf{u}^1) &\leq 0 && \text{(2b)}\\ \sigma_n^1(\mathbf{u}^1)([\![u]\!]_n^1 - g_n^1) &= 0 && \text{(2c)} \end{aligned}\right\} \quad \text{on } \Gamma_C^1,$$

$$\left.\begin{aligned} [\![u]\!]_n^2 &\leq g_n^2 && \text{(3a)}\\ \sigma_n^2(\mathbf{u}^2) &\leq 0 && \text{(3b)}\\ \sigma_n^2(\mathbf{u}^2)([\![u]\!]_n^2 - g_n^2) &= 0 && \text{(3c)} \end{aligned}\right\} \quad \text{on } \Gamma_C^2.$$

Let $s^i \in L^2(\Gamma_C^i)$, $s^i \geq 0$, $[\![\mathbf{u}]\!]_t^1 = \mathbf{u}_t^1 - \mathbf{u}_t^2 \circ \Pi^1$ and $[\![\mathbf{u}]\!]_t^2 = \mathbf{u}_t^2 - \mathbf{u}_t^1 \circ \Pi^2 = -[\![\mathbf{u}]\!]_t^1 \circ \Pi^2$.
The Tresca friction condition on Γ_C^1 and Γ_C^2 reads:

$$\begin{cases} \|\boldsymbol{\sigma}_t^i(\mathbf{u}^i)\| \leq s^i & \text{if } [\![\mathbf{u}]\!]_t^i = 0, \\ \boldsymbol{\sigma}_t^i(\mathbf{u}^i) = -s^i \dfrac{[\![\mathbf{u}]\!]_t^i}{\|[\![\mathbf{u}]\!]_t^i\|} & \text{otherwise,} \end{cases} \tag{4}$$

where $\|\cdot\|$ stands for the euclidean norm in $\mathbb{R}^{d-1}$.

Remark 1.2. *In the frictionless contact case this condition is simply replaced by* $\boldsymbol{\sigma}_t^i = 0$.

Finally, we need to consider the second Newton law between the two bodies:

$$\begin{cases} \displaystyle\int_{\gamma_C^1} \sigma_n^1(\mathbf{u}^1)\mathrm{d}s - \int_{\gamma_C^2} \sigma_n^2(\mathbf{u}^2)\mathrm{d}s = 0, \\ \displaystyle\int_{\gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\mathrm{d}s + \int_{\gamma_C^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2)\mathrm{d}s = 0, \end{cases}$$

where γ_C^1 is any subset of Γ_C^1 and $\gamma_C^2 = \Pi^1(\gamma_C^1)$. Mapping all terms on γ_C^1 allows writing:

$$\begin{cases} \displaystyle\int_{\gamma_C^1} \sigma_n^1(\mathbf{u}^1) - J^1\sigma_n^2(\mathbf{u}^2 \circ \Pi^1)\mathrm{d}s = 0, \\ \displaystyle\int_{\gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1) + J^1\boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1)\mathrm{d}s = 0, \end{cases} \qquad \forall \gamma_C^1 \subset \Gamma_C^1$$

so we obtain:

$$\begin{cases} \sigma_n^1(\mathbf{u}^1) - J^1\sigma_n^2(\mathbf{u}^2 \circ \Pi^1) = 0, \\ \boldsymbol{\sigma}_t^1(\mathbf{u}^1) + J^1\boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1) = 0, \end{cases} \quad \text{on } \Gamma_C^1. \tag{5}$$

Remark 1.3. *: A similar condition holds on* Γ_c^2*:*

$$\begin{cases} \sigma_n^2(\mathbf{u}^2) - J^2\sigma_n^1(\mathbf{u}^1 \circ \Pi^2) = 0, \\ \boldsymbol{\sigma}_t^2(\mathbf{u}^2) + J^2\boldsymbol{\sigma}_t^1(\mathbf{u}^1 \circ \Pi^2) = 0. \end{cases}$$

It is important to mention that, due to second Newton law, we need to fix s^1 and s^2 such that: $-s^1 \dfrac{[\![\mathbf{u}]\!]_t^1}{\|[\![\mathbf{u}]\!]_t^1\|} = \boldsymbol{\sigma}_t^1(\mathbf{u}^1) = -J^1\boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1) = J^1 s^2 \dfrac{[\![\mathbf{u}]\!]_t^2 \circ \Pi^1}{\|[\![\mathbf{u}]\!]_t^2 \circ \Pi^1\|} = -J^1 s^2 \dfrac{[\![\mathbf{u}]\!]_t^1}{|[\![\mathbf{u}]\!]_t^1|}$.
And so:

$$s^1 = J^1 s^2. \tag{6}$$

1.2 Variational formulation using Nitsche's method

In this section, we establish the weak formulation of Problem (1)–(5) using Nitsche's method and the unbiased writing of the contact and the friction conditions given in Section 1.1.
As in [7], we introduce an additional parameter θ. This generalization will allow several variants, depending on the value of θ. The symmetric case is obtained when $\theta = 1$. The advantage of the symmetric formulation is that it derives from a potential of energy (see 1.3). These features are lost when $\theta \neq 1$. Nevertheless the variants $\theta = -1$ and 0 presents some other advantages, mostly from the numerical viewpoint. In particular, the case $\theta = 0$ involves a reduced quantity of terms, which makes it easier to implement and to extend to contact problems involving non-linear elasticity. Also, for $\theta = -1$, the well-posedness of the discrete formulation and the optimal convergence are preserved irrespectively of the value of the Nitsche parameter γ^i.

First, we introduce the Hilbert space

$$\mathbf{V} = \Big\{\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in H^1(\Omega^1)^d \times H^1(\Omega^2)^d : \quad \mathbf{v}^1 = 0 \text{ on } \Gamma_D^1 \text{ and } \mathbf{v}^2 = 0 \text{ on } \Gamma_D^2\Big\}.$$

Let $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$ be the solution of the contact problem in its strong form (1)–(5). We assume that $\mathbf{u}$ is sufficiently regular so that all the following calculations make sense.
The derivation of a Nitsche-based method comes from a reformulation of the contact conditions (2a)-(2b)-(2c) (see for instance [5] and [7]). This reformulation comes from the augmented Lagrangian formulation of contact problems. The contact conditions (2a)-(2b)-(2c) are equivalent to the equation (7) for a given positive function γ^i :

$$\sigma_n^i(\mathbf{u}^i) = -\frac{1}{\gamma^i}[(\llbracket u \rrbracket_n^i - g_n^i) - \gamma^i \sigma_n^i(\mathbf{u}^i)]_+, \tag{7}$$

where the notation $[\cdot]_+$ refers to the the positive part of a scalar quantity $\mathbf{a} \in \mathbb{R}$. Similarly, in [4], the Tresca friction condition is equivalent to the equation

$$\boldsymbol{\sigma}_t(\mathbf{u}^i) = -\frac{1}{\gamma^i}[\llbracket \mathbf{u} \rrbracket_t^i - \gamma^i \boldsymbol{\sigma}(\mathbf{u}^i)]_{\gamma s^i}, \tag{8}$$

where, for any $\alpha \in \mathbb{R}^+$, the notation $[\cdot]_\alpha$ refers to the orthogonal projection onto $\mathcal{B}(0, \alpha) \subset \mathbb{R}^{d-1}$, the closed ball centered at the origin and of radius α. In what follows some properties of the positive part and the projection are mentioned. Those properties will be useful in the analysis of the method.
Since $a \leq [a]_+$ and $a[a]_+ = [a]_+^2 \ \forall a \in \mathbb{R}$, we can write that for all $a, b \in \mathbb{R}$:

$$\begin{aligned} ([a]_+ - [b]_+)(a - b) &= a[a]_+ + b[b]_+ - b[a]_+ - a[b]_+ \\ &\leq [a]_+^2 + [b]_+^2 - 2[a]_+[b]_+ \\ &= ([a]_+ - [b]_+)^2. \end{aligned} \tag{9}$$

We note, also, the following classical property for a projection for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$:

$$(\mathbf{y} - \mathbf{x}).([\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha) \geq \|[\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha\|^2. \tag{10}$$

From the Green formula and equations (1), we get for every $\mathbf{v} \in \mathbf{V}$:

$$\int_{\Omega^1} \boldsymbol{\sigma}^1(\mathbf{u}^1) : \boldsymbol{\varepsilon}(\mathbf{v}^1) \mathrm{d}\Omega + \int_{\Omega^2} \boldsymbol{\sigma}^2(\mathbf{u}^2) : \boldsymbol{\varepsilon}(\mathbf{v}^2) \mathrm{d}\Omega = \int_{\Omega^1} \mathbf{f}^1 \cdot \mathbf{v}^1 \mathrm{d}\Omega + \int_{\Omega^2} \mathbf{f}^2 \cdot \mathbf{v}^2 \mathrm{d}\Omega$$
$$+ \int_{\Gamma_N^1} \mathbf{t}^1 \cdot \mathbf{v}^1 \mathrm{d}\Gamma + \int_{\Gamma_N^2} \mathbf{t}^2 \cdot \mathbf{v}^2 \mathrm{d}\Gamma + \int_{\Gamma_C^1} \boldsymbol{\sigma}^1(\mathbf{u}^1)\mathbf{n}^1 \cdot \mathbf{v}^1 \mathrm{d}\Gamma + \int_{\Gamma_C^2} \boldsymbol{\sigma}^2(\mathbf{u}^2)\mathbf{n}^2 \cdot \mathbf{v}^2 \mathrm{d}\Gamma.$$

We define

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega^1} \boldsymbol{\sigma}^1(\mathbf{u}^1) : \boldsymbol{\varepsilon}(\mathbf{v}^1) \mathrm{d}\Omega + \int_{\Omega^2} \boldsymbol{\sigma}^2(\mathbf{u}^2) : \boldsymbol{\varepsilon}(\mathbf{v}^2) \mathrm{d}\Omega, \\ &\text{and} \\ L(\mathbf{v}) &= \int_{\Omega^1} \mathbf{f}^1 \cdot \mathbf{v}^1 \mathrm{d}\Omega + \int_{\Omega^2} \mathbf{f}^2 \cdot \mathbf{v}^2 \mathrm{d}\Omega + \int_{\Gamma_N^1} \mathbf{t}^1 \cdot \mathbf{v}^1 \mathrm{d}\Gamma + \int_{\Gamma_N^2} \mathbf{t}^2 \cdot \mathbf{v}^2 \mathrm{d}\Gamma. \end{aligned}$$

So, there holds:

$$a(\mathbf{u}, \mathbf{v}) - \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) v_n^1 \mathrm{d}\Gamma - \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) v_n^2 \mathrm{d}\Gamma - \int_{\Gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot \mathbf{v}_t^1 \mathrm{d}\Gamma - \int_{\Gamma_C^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2) \cdot \mathbf{v}_t^2 \mathrm{d}\Gamma = L(\mathbf{v}).$$

Using condition (5) we can write

$$a(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1}(\sigma_n^1(\mathbf{u}^1) + J^1\sigma_n^2(\mathbf{u}^2\circ\Pi^1))v_n^1 \mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}(\sigma_n^2(\mathbf{u}^2) + J^2\sigma_n^1(\mathbf{u}^1\circ\Pi^2))v_n^2 \mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^1}(\boldsymbol{\sigma}_t^1(\mathbf{u}^1) - J^1\boldsymbol{\sigma}_t^2(\mathbf{u}^2\circ\Pi^1))\cdot\mathbf{v}_t^1\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}(\boldsymbol{\sigma}_t^2(\mathbf{u}^2) - J^2\boldsymbol{\sigma}_t^1(\mathbf{u}^1\circ\Pi^2))\cdot\mathbf{v}_t^2\mathrm{d}\Gamma = L(\mathbf{v}).$$

So, using the property $\int_{\Gamma_C^1} J^1 f \mathrm{d}\Gamma = \int_{\Gamma_C^2} f\circ\Pi^2\mathrm{d}\Gamma$, we have

$$a(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1}\sigma_n^1(\mathbf{u}^1)v_n^1\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\sigma_n^1(\mathbf{u}^1)(v_n^2\circ\Pi^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}\sigma_n^2(\mathbf{u}^2)v_n^2\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^2}\sigma_n^2(\mathbf{u}^2)(v_n^1\circ\Pi^2)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot\mathbf{v}_t^1 + \frac{1}{2}\int_{\Gamma_C^1}\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot(\mathbf{v}_t^2\circ\Pi^1)\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^2}\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\mathbf{v}_t^2 + \frac{1}{2}\int_{\Gamma_C^2}\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot(\mathbf{v}_t^1\circ\Pi^2)\mathrm{d}\Gamma = L(\mathbf{v}).$$

This leads to:

$$a(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1}\sigma_n^1(\mathbf{u}^1)(v_n^1 + v_n^2\circ\Pi^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}\sigma_n^2(\mathbf{u}^2)(v_n^2 + v_n^1\circ\Pi^2)\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^1}\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot(\mathbf{v}_t^1 - \mathbf{v}_t^2\circ\Pi^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot(\mathbf{v}_t^2 - \mathbf{v}_t^1\circ\Pi^2)\mathrm{d}\Gamma = L(\mathbf{v}).$$

With the writings, for $\theta\in\mathbb{R}$:

$$\begin{cases} v_n^1 + v_n^2\circ\Pi^1 = (v_n^1 + v_n^2\circ\Pi^1 - \theta\gamma^1\sigma_n^1(\mathbf{v}^1)) + \theta\gamma^1\sigma_n^1(\mathbf{v}^1)\\ v_n^2 + v_n^1\circ\Pi^2 = (v_n^2 + v_n^1\circ\Pi^2 - \theta\gamma^2\sigma_n^2(\mathbf{v}^2)) + \theta\gamma^2\sigma_n^2(\mathbf{v}^2)\end{cases}$$
$$\begin{cases} \mathbf{v}_t^1 - \mathbf{v}_t^2\circ\Pi^1 = (\mathbf{v}_t^1 - \mathbf{v}_t^2\circ\Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1)) + \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1)\\ \mathbf{v}_t^2 - \mathbf{v}_t^1\circ\Pi^2 = (\mathbf{v}_t^2 - \mathbf{v}_t^1\circ\Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2)) + \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\end{cases}$$

we obtain:
(11)

$$a(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1}\theta\gamma^1\sigma_n^1(\mathbf{u}^1)\sigma_n^1(\mathbf{v}^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}\theta\gamma^2\sigma_n^2(\mathbf{u}^2)\sigma_n^2(\mathbf{v}^2)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot\boldsymbol{\sigma}_t^1(\mathbf{v}^1)\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^2}\theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\sigma_n^1(\mathbf{u}^1)(v_n^1 + v_n^2\circ\Pi^1 - \theta\gamma^1\sigma_n^1(\mathbf{v}^1))\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^2}\sigma_n^2(\mathbf{u}^2)(v_n^2 + v_n^1\circ\Pi^2 - \theta\gamma^2\sigma_n^2(\mathbf{v}^2))\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot(\mathbf{v}_t^1 - \mathbf{v}_t^2\circ\Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1))\mathrm{d}\Gamma$$
$$-\frac{1}{2}\int_{\Gamma_C^2}\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\big(\mathbf{v}_t^2 - \mathbf{v}_t^1\circ\Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\big)\mathrm{d}\Gamma = L(\mathbf{v}).$$

Let us define:

$$(12)\quad \begin{aligned} P^i_{n,\gamma^i}(\mathbf{u}) &= [\![u]\!]^i_n - \gamma^i\boldsymbol{\sigma}^i_n(\mathbf{u}^i) - g^i_n, & \mathbf{P}^i_{t,\gamma^i}(\mathbf{u}) &= [\![\mathbf{u}]\!]^i_t - \gamma^i\boldsymbol{\sigma}^i_t(\mathbf{u}^i),\\ P^i_{n,\theta\gamma^i}(\mathbf{v}) &= [\![v]\!]^i_n - \theta\gamma^i\sigma^i_n(\mathbf{v}^i), & \mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{v}) &= [\![\mathbf{v}]\!]^i_t - \theta\gamma^i\boldsymbol{\sigma}^i_t(\mathbf{v}^i)\end{aligned}$$

$$
\begin{aligned}
\text{and } A_\theta(\mathbf{u},\mathbf{v}) &= \mathbf{a}(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1} \theta\gamma^1\sigma_n^1(\mathbf{u}^1)\sigma_n^1(\mathbf{v}^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2} \theta\gamma^2\sigma_n^2(\mathbf{u}^2)\sigma_n^2(\mathbf{v}^2)\mathrm{d}\Gamma \\
&\quad -\frac{1}{2}\int_{\Gamma_C^1} \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot\boldsymbol{\sigma}_t^1(\mathbf{v}^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2} \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\mathrm{d}\Gamma \\
&= \mathbf{a}(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1} \theta\gamma^1\boldsymbol{\sigma}^1(\mathbf{u}^1)\mathbf{n}\cdot\boldsymbol{\sigma}^1(\mathbf{v}^1)\mathbf{n}\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2} \theta\gamma^2\boldsymbol{\sigma}^2(\mathbf{u}^2)\mathbf{n}\cdot\boldsymbol{\sigma}^2(\mathbf{v}^2)\mathbf{n}\mathrm{d}\Gamma.
\end{aligned}
$$

Now we insert the expressions (7) of $\sigma_n^i(u^i)$ and (8) of $\boldsymbol{\sigma}_t^i(\mathbf{u}^i)$ in (11) and the variational problem could be formally written as follows:

(13)

$$
\left\{
\begin{array}{l}
\text{Find a sufficiently regular } \mathbf{u}\in\mathbf{V} \text{ such that for all sufficiently regular } \mathbf{v}\in\mathbf{V}, \\[2ex]
A_\theta(\mathbf{u},\mathbf{v}) + \dfrac{1}{2}\displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma^1}[P^1_{n,\gamma^1}(\mathbf{u})]_+ P^1_{n,\theta\gamma^1}(\mathbf{v})\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2}[P^2_{n,\gamma^2}(\mathbf{u})]_+ P^2_{n,\theta\gamma^2}(\mathbf{v})\mathrm{d}\Gamma \\
+\dfrac{1}{2}\displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma^1}[\mathbf{P}^1_{t,\gamma^1}(\mathbf{u})]_{\gamma^1 s^1}\cdot\mathbf{P}^1_{t,\theta\gamma^1}(\mathbf{v})\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2}[\mathbf{P}^2_{t,\gamma^2}(\mathbf{u})]_{\gamma^2 s^2}\cdot\mathbf{P}^2_{t,\theta\gamma^2}(\mathbf{v})\mathrm{d}\Gamma = L(\mathbf{v}).
\end{array}
\right.
$$

Remark 1.4. *In the frictionless contact case the formulation reads:*

$$
\left\{
\begin{array}{l}
\textit{Find } \mathbf{u}\in\mathbf{V} \textit{ such that } \forall\mathbf{v}\in\mathbf{V} \\[2ex]
A_\theta(\mathbf{u},\mathbf{v}) + \dfrac{1}{2}\displaystyle\int_{\Gamma_C^1} \frac{1}{\gamma^1}[P^1_{n,\gamma^1}(\mathbf{u})]_+ P^1_{n,\theta\gamma^1}(\mathbf{v})\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2}[P^2_{n,\gamma^2}(\mathbf{u})]_+ P^2_{n,\theta\gamma^2}(\mathbf{v})\mathrm{d}\Gamma = L(\mathbf{v}).
\end{array}
\right.
$$

.

1.3 Derivation of the method from a potential

In this section we show, through a formal demonstration, that the method derives from a potential in the frictional symmetric ($\theta = 1$) case. Let us define the potential:

$$
J(\mathbf{u}) = \boldsymbol{\varepsilon}_\Omega + \sum_{i=1}^{2}(\boldsymbol{\varepsilon}_n^i + \boldsymbol{\varepsilon}_t^i),
$$

with:

$$
\begin{aligned}
\boldsymbol{\varepsilon}_\Omega &= \frac{1}{2}a(\mathbf{u},\mathbf{u}) - \sum_{i=1}^{2}\Big(\frac{1}{4}\int_{\Gamma_C^i}\gamma^i(\sigma_n^i(\mathbf{u}^i))^2 + \frac{1}{4}\int_{\Gamma_C^i}\gamma^i\|\boldsymbol{\sigma}_t^i(\mathbf{u}^i)\|^2\mathrm{d}\Gamma\Big) - L(\mathbf{u}) \\
&= \frac{1}{2}A_1(\mathbf{u},\mathbf{u}) - L(\mathbf{u}); \\
\boldsymbol{\varepsilon}_n^i &= \frac{1}{4}\int_{\Gamma_C^i}\frac{1}{\gamma^i}[P^i_{n,\gamma^i}(\mathbf{u})]_+^2\mathrm{d}\Gamma; \\
\boldsymbol{\varepsilon}_t^i &= \frac{1}{4}\int_{\Gamma_C^i}\frac{1}{\gamma^i}\|\mathbf{P}^i_{t,\gamma^i}(\mathbf{u})\|^2\mathrm{d}\Gamma - \frac{1}{4}\int_{\Gamma_C^i}\frac{1}{\gamma^i}\|\mathbf{P}^i_{t,\gamma^i}(\mathbf{u}) - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{u})]_{\gamma^i s^i}\|^2\mathrm{d}\Gamma.
\end{aligned}
$$

We compute now the derivative of this potential. We have:

$$
D\boldsymbol{\varepsilon}_\Omega(\mathbf{u})[\mathbf{v}] \;= A_1(\mathbf{u},\mathbf{v}) - L(\mathbf{v}) \quad (L \text{ is linear and } A_\theta \text{ is bilinear}),
$$

$$\begin{aligned} D\boldsymbol{\varepsilon}_n^i(u)[\mathbf{v}] &= \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u})]_+ D\big([P_{n,\gamma^i}^i(\mathbf{u})]_+\big)[\mathbf{v}]\mathrm{d}\Gamma \\ &= \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u})]_+ H\big(P_{n,\gamma^i}^i(\mathbf{u})\big)\big(D\big(P_{n,\gamma^i}^i(\mathbf{u})\big)[\mathbf{v}]\mathrm{d}\Gamma, \end{aligned}$$

where H is the Heaviside step function. Using the equalities: $H(\varphi(X))[\varphi(X)]_+ = [\varphi(X)]_+$ and $D\big(P_{n,\gamma^i}^i(\mathbf{u})\big)[\mathbf{v}] = P_{n,\gamma^i}^i(\mathbf{v})$ (since P_{n,γ^i}^i is linear), we get:

$$D\boldsymbol{\varepsilon}_n^i(u)[\mathbf{v}] = \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u})]_+ P_{n,\gamma^i}^i(\mathbf{v})\mathrm{d}\Gamma.$$

$$\begin{aligned} D\boldsymbol{\varepsilon}_t^i(u)[\mathbf{v}] &= \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})\cdot\mathbf{P}_{t,\gamma^i}^i(\mathbf{v})\mathrm{d}\Gamma \\ &\quad -\frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}(\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i})\cdot(\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}) - D\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)[\mathbf{v}])\mathrm{d}\Gamma \end{aligned}$$

$$\begin{cases} \text{if } \|\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})\| \le \gamma^i s^i, \text{then } \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i} = 0 \\ \text{if } \|\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})\| > \gamma^i s^i, \text{then } D\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i} \text{is tangential to } \mathcal{B}(0,\gamma^i s^i) \text{ and} \\ D\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\cdot(\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}) - [\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}) = 0. \end{cases}$$

So, in both cases we have:

$$D\boldsymbol{\varepsilon}_t^i(u)[\mathbf{v}] = \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\cdot\mathbf{P}_{t,\gamma^i}^i(\mathbf{v})\mathrm{d}\Gamma$$

so, if we consider the first order optimality condition $D\boldsymbol{\varepsilon}(\mathbf{u})[\mathbf{v}] = 0 \forall \mathbf{v}\in\mathbf{V}$, we get:

$$A_1(\mathbf{u},\mathbf{v}) + \sum_{i=1}^{2}\Big(\frac{1}{2}\int_{\Gamma_C^i}\frac{1}{\gamma^i}[P_{n,\gamma^i}^i(\mathbf{u})]_+ P_{n,\gamma^i}^i(\mathbf{v})\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^i}\frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\cdot\mathbf{P}_{t,\gamma^i}^i(\mathbf{v})\mathrm{d}\Gamma\Big) = L(\mathbf{v}).$$

1.4 Strong-weak formulation equivalence

In this section, we are going to establish the formal equivalence between (13) and (1)-(5). Since the construction of (13) is quite elaborated and consists in particular in the splitting of the contact terms into two parts, this step is necessary to ensure the coherence of the formulation.

Theorem 1.5. *Let* $\mathbf{u} = (u^1, u^2)$ *be a sufficiently regular solution to the problem (13), then* $\mathbf{u}$ *solves the problem (1)-(5) for all* $\theta\in\mathbb{R}$.

Proof. Let $\mathbf{u} = (u^1, u^2)$ be a sufficiently regular solution to the problem (13). Using the definitions of $A_\theta, P_{\gamma^i}^i(\mathbf{u})$ and $P_{\theta\gamma^i}^i(\mathbf{v})$, we obtain:

$$\begin{aligned} &\mathbf{a}(\mathbf{u},\mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1}\theta\gamma^1\sigma_n^1(\mathbf{u}^1)\sigma_n^1(\mathbf{v}^1)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2}\theta\gamma^2\sigma_n^2(\mathbf{u}^2)\sigma_n^2(\mathbf{v}^2)\mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot\boldsymbol{\sigma}_t^1(\mathbf{v}^1)\mathrm{d}\Gamma \\ &-\frac{1}{2}\int_{\Gamma_C^2}\theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma^1}[[\![u]\!]_n^1 - g_n^1 - \gamma^1\sigma_n^1(\mathbf{u}^1)]_+(v_n^1 + v_n^2\circ\Pi^1 - \theta\gamma^1\sigma_n^1(\mathbf{v}^1))\mathrm{d}\Gamma \\ &+\frac{1}{2}\int_{\Gamma_C^2}\frac{1}{\gamma^2}[[\![u]\!]_n^2 - g_n^2 - \gamma^2\sigma_n^2(\mathbf{u}^2)]_+(v_n^2 + v_n^1\circ\Pi^2 - \theta\gamma^2\sigma_n^2(\mathbf{v}^2))\mathrm{d}\Gamma \\ &+\frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma^1}[[\![\mathbf{u}]\!]_t^1 - \gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)]_{\gamma^1 s^1}\cdot(\mathbf{v}_t^1 - \mathbf{v}_t^2\circ\Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1))\mathrm{d}\Gamma \\ &+\frac{1}{2}\int_{\Gamma_C^2}\frac{1}{\gamma^2}[[\![\mathbf{u}]\!]_t^2 - \gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2)]_{\gamma^2 s^2}\cdot\big(\mathbf{v}_t^2 - \mathbf{v}_t^1\circ\Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\big)\mathrm{d}\Gamma = L(\mathbf{v}). \end{aligned}$$

Using Green's formula we can write

$$\begin{aligned}\mathbf{a}(\mathbf{u},\mathbf{v}) &= -\int_{\Omega^1}\mathbf{div}\boldsymbol{\sigma}^1(\mathbf{u}^1)\cdot\mathbf{v}^1 d\Omega - \int_{\Omega^2}\mathbf{div}\boldsymbol{\sigma}^2(\mathbf{u}^2)\cdot\mathbf{v}^2 d\Omega \\ &+\int_{\partial\Omega^1}\boldsymbol{\sigma}^1(\mathbf{u}^1)\mathbf{n}^1\cdot\mathbf{v}^1 d\Gamma + \int_{\partial\Omega^2}\boldsymbol{\sigma}^2(\mathbf{u}^2)\mathbf{n}^2\cdot\mathbf{v}^2 d\Gamma.\end{aligned}$$

If we take $\mathbf{v} = (\mathbf{v}^1, 0)$ with $\mathbf{v}^1 = 0$ on $\partial\Omega^1$, we obtain:

$$\int_{\Omega^1}\mathbf{div}\boldsymbol{\sigma}^1(\mathbf{u}^1)\cdot\mathbf{v}^1 d\Omega = \int_{\Omega^1}\mathbf{f}^1\cdot\mathbf{v}^1 d\Omega \ \forall\mathbf{v}^1,$$

which yields (1a) for i=1. In the same way we establish (1a) for i=2.
To establish (2),(3),(4) and (5), we consider a displacement field $\mathbf{v}$ that vanishes on the boundary except on the contact surfaces where $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$. Then (13) and (1a) gives
(14)

$$\begin{aligned}&\int_{\Gamma_C^1}\sigma_n^1(\mathbf{u}^1)v_n^1 d\Gamma + \int_{\Gamma_C^1}\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot\mathbf{v}_t^1 d\Gamma + \int_{\Gamma_C^2}\sigma_n^2(\mathbf{u}^2)v_n^2 d\Gamma + \int_{\Gamma_C^2}\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\mathbf{v}_t^2 d\Gamma \\ &-\frac{1}{2}\int_{\Gamma_C^1}\theta\gamma^1\sigma_n^1(\mathbf{u}^1)\sigma_n^1(\mathbf{v}^1)d\Gamma - \frac{1}{2}\int_{\Gamma_C^2}\theta\gamma^2\sigma_n^2(\mathbf{u}^2)\sigma_n^2(\mathbf{v}^2)d\Gamma - \frac{1}{2}\int_{\Gamma_C^1}\theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot\boldsymbol{\sigma}_t^1(\mathbf{v}^1)d\Gamma \\ &-\frac{1}{2}\int_{\Gamma_C^2}\theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot\boldsymbol{\sigma}_t^2(\mathbf{v}^2)d\Gamma + \frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma^1}[[\![u]\!]_n^1 - g_n^1 - \gamma^1\sigma_n^1(\mathbf{u}^1)]_+(v_n^1 + v_n^2\circ\Pi^1 - \theta\gamma^1\sigma_n^1(\mathbf{v}^1))d\Gamma \\ &+\frac{1}{2}\int_{\Gamma_C^2}\frac{1}{\gamma^2}[[\![u]\!]_n^2 - g_n^2 - \gamma^2\sigma_n^2(\mathbf{u}^2)]_+(v_n^2 + v_n^1\circ\Pi^2 - \theta\gamma^2\sigma_n^2(\mathbf{v}^2))d\Gamma \\ &+\frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma^1}[[\![\mathbf{u}]\!]_t^1 - \gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)]_{\gamma^1 s^1}\cdot(\mathbf{v}_t^1 - \mathbf{v}_t^2\circ\Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}^1))d\Gamma \\ &+\frac{1}{2}\int_{\Gamma_C^2}\frac{1}{\gamma^2}[[\![\mathbf{u}]\!]_t^2 - \gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2)]_{\gamma^2 s^2}\cdot\big(\mathbf{v}_t^2 - \mathbf{v}_t^1\circ\Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}^2)\big)d\Gamma = 0.\end{aligned}$$

We need to discuss two cases: $\theta \neq 0$ and $\theta = 0$.

Case 1 $\theta \neq 0$:
In (14), let us consider $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that:

$$(15)\qquad \begin{cases}\mathbf{v}^1 = 0 \text{ and } \boldsymbol{\sigma}_t^1(\mathbf{v}^1) = 0\ ,\ \sigma_n^1(\mathbf{v}^1)\neq 0 & \text{on } \Gamma_C^1 \text{ and} \\ \mathbf{v}^2 = 0 \text{ and } \boldsymbol{\sigma}^2(\mathbf{v}^2)\mathbf{n}^2 = 0 & \text{on } \Gamma_C^2,\end{cases}$$

so,

$$\frac{\theta}{2}\int_{\Gamma_C^1}\Big([[\![u]\!]_n^1 - g_n^1 - \gamma^1\sigma_n^1(\mathbf{u}^1)]_+ + \gamma^1\sigma_n^1(\mathbf{u}^1)\Big)\sigma_n^1(\mathbf{v}^1)d\Gamma = 0 \qquad \forall\mathbf{v} \text{ satisfying (15)}.$$

Then:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{\gamma^1}[[\![u]\!]_n^1 - g_n^1 - \gamma^1\sigma_n^1(\mathbf{u}^1)]_+,$$

which implies (2). Arguing in the same way we obtain (3) and the friction conditions (4).

Remark 1.6. *It is easy to show that* $\mathbf{v}$ *satisfying (15) can be built by considering* $\mathbf{s}(\mathbf{x})$ *the curvilinear coordinate system on the boundary* Γ_C *and* $d(x)$ *the signed distance to* Γ_C*. Then, for* $\mathbf{g}$ *a given vector field of* $\mathbb{R}^d$*,* $\mathbf{u}(\mathbf{x}) = B^{-1}(\mathbf{s}(\mathbf{x}))\mathbf{g}(\mathbf{s}(\mathbf{x}))d(\mathbf{x})$ *satisfies* $\mathbf{u}(\mathbf{x}) = 0$ *and* $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g}$ *on* Γ_C*, with* $B_{il} = A_{ijkl}n_k n_j$*, A being the elasticity tensor.*

To obtain the second Newton law, we use Nitsche's writing of (2) and (3) in (14) with: $\mathbf{v}_t = 0$ and $\boldsymbol{\sigma}_t = 0$ and $v_n^2 = -v_n^1 \circ \Pi^2$:

$$\int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) v_n^1 \mathrm{d}\Gamma - \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) v_n^1 \circ \Pi^2 \mathrm{d}\Gamma = 0 \qquad \forall v_n^1.$$

Then:

$$\int_{\Gamma_C^1} [\sigma_n^1(\mathbf{u}^1) - J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)] v_n^1 \mathrm{d}\Gamma = 0 \qquad \forall v_n^1.$$

For $v_n^1 = v_n^2 = 0$ and $\mathbf{v}_t^2 = \mathbf{v}_t^1 \circ \Pi^2$ and using (4) in (14) ,we have similary

$$\int_{\Gamma_C^1} [\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + J^1 \boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1)] \cdot \mathbf{v}_t^1 \mathrm{d}\Gamma = 0 \qquad \forall \mathbf{v}_t^1,$$

and we have (5).

Case 2 $\theta = 0$**:**
Let us take $\mathbf{v}_t^1 = \mathbf{v}_t^2 = 0$ and $v_n^2 = -v_n^1 \circ \Pi^2$, $v_n^1 = -v_n^2 \circ \Pi^1$, then (14) reads:

$$\int_{\Gamma_C^1} [\sigma_n^1(\mathbf{u}^1) - J^1 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)] v_n^1 \mathrm{d}\Gamma = 0 \qquad \forall v_n^1.$$

Let us take,now $v_n^1 = v_n^2 = 0$ and $\mathbf{v}_t^2 = \mathbf{v}_t^1 \circ \Pi^2$, $\mathbf{v}_t^1 = \mathbf{v}_t^2 \circ \Pi^1$, then (14) reads:

$$\int_{\Gamma_C^1} [\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + J^1 \boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1)] \cdot \mathbf{v}_t^1 \mathrm{d}\Gamma = 0 \qquad \forall \mathbf{v}_t^1,$$

and we have (5).
Let $\mathbf{v}^2 = 0$ on Γ_C^2. Taking $\mathbf{v}_t^1 = 0$, we get:

$$\int_{\Gamma_C^1} \Big[\sigma_n^1(\mathbf{u}^1) + \frac{1}{2\gamma^1}[[\![u]\!]_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1)]_+ + J^1 \frac{1}{2\gamma^2}[[\![u]\!]_n^2 \circ \Pi^1 - g_n^2 \circ \Pi^1 - \gamma^2 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)]_+\Big] v_n^1 \mathrm{d}\Gamma = 0 \ \forall v_n^1.$$

Then:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{2}\Big[\frac{1}{\gamma^1}[[\![u]\!]_n^1 - g_n^1 - \gamma^1 \sigma_n^1(\mathbf{u}^1)]_+ + \frac{J^1}{\gamma^2}[[\![u]\!]_n^1 - g_n^1 - \gamma^2 \sigma_n^2(\mathbf{u}^2 \circ \Pi^1)]_+\Big].$$

Since $J^1 > 0$, $\sigma_n^1(\mathbf{u}^1) \leq 0$ and so we obtain (2b. The second Newton law (5) yields:

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{2}\Big[\frac{1}{\gamma^1}[([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1)]_+ + [\frac{J^1}{\gamma^2}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1)]_+\Big]. \tag{16}$$

We discuss both cases:
If $\sigma_n^1(\mathbf{u}^1) = 0$:

$$\frac{1}{2}(\frac{1}{\gamma^1} + \frac{J^1}{\gamma^2})[[\![u]\!]_n^1 - g_n^1]_+ = 0 \text{ then } [\![u]\!]_n^1 \leq g_n^1.$$

If $\sigma_n^1(\mathbf{u}^1) < 0$:

$$\frac{1}{\gamma^1}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0 \qquad \text{or} \qquad \frac{J^1}{\gamma^2}(([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1)) > 0 \qquad \text{or both .}$$

1. If we suppose first that: $\dfrac{1}{\gamma^1}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$ and $\dfrac{J^1}{\gamma^2}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$, the equation (16) holds :

$$\sigma_n^1(\mathbf{u}^1) = -\frac{1}{2}[(\frac{1}{\gamma^1} + \frac{J^1}{\gamma^2})([\![u]\!]_n^1 - g_n^1) - 2\sigma_n^1(\mathbf{u}^1)] \qquad \text{then} \qquad [\![\mathbf{u}]\!]_n^1 = g_n^1.$$

2. If now there only holds $\dfrac{1}{\gamma^1}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$ and $\dfrac{J^1}{\gamma^2}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) = 0$, we can write (16):

$$\begin{aligned} \sigma_n^1(\mathbf{u}^1) &= -\frac{1}{2\gamma^1}([\![u]\!]_n^1 - g_n^1) + \frac{1}{2}\sigma_n^1(\mathbf{u}^1)). \\ \text{So } \sigma_n^1(\mathbf{u}^1) &= -\frac{1}{\gamma}([\![u]\!]_n^1 - g_n^1). \end{aligned}$$

   Then, since $\sigma_n(\mathbf{u}^1) < 0$: $[\![u]\!]_n^1 > g_n^1$. But in this case,

$$\frac{J^1}{\gamma^2}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0,$$

   and this contradicts the assumption $\dfrac{J^1}{\gamma^2}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) = 0$. So, this case is absurd.
   In a similar way we get contradiction for the case $\dfrac{J^1}{\gamma^2}([\![u]\!]_n^1 - g_n^1) - \sigma_n^1(\mathbf{u}^1) > 0$.

To conclude, we establish that: if $\sigma_n^1(\mathbf{u}^1) = 0$, $[\![u]\!]_n^1 \leq g_n^1$ and if $\sigma_n^1(\mathbf{u}^1) < 0$, $[\![u]\!]_n^1 = g_n^1$; and this is equivalent to (2a) and (2c).

We suppose, now, that $v_n^1 = 0$ and $\mathbf{v}^2 = 0$. We get:

$$\int_{\Gamma_C^1} \Big[\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + \frac{1}{2\gamma^1}[[\![\mathbf{u}]\!]_t^1 - \gamma^1\boldsymbol{\sigma}_t^1(\mathbf{u}^1)]_{\gamma^1 s^1} - \frac{J^1}{2\gamma^2}[[\![\mathbf{u}]\!]_t^2 \circ \Pi^1 - \gamma^2\boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1)]_{\gamma^2 s^2}\Big] \cdot \mathbf{v}_t^1 \mathrm{d}\Gamma = 0 \ \forall \mathbf{v}_t^1.$$

Then, using the property: $\forall\gamma > 0$, $[x]_{\gamma s} = \gamma\big[\frac{x}{\gamma}\big]_s$, it yields:

$$\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + \frac{1}{2}\Big[\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\Big]_{s^1} - \frac{J^1}{2}\Big[\frac{[\![\mathbf{u}]\!]_t^2 \circ \Pi^1}{\gamma^2} - \boldsymbol{\sigma}_t^2(\mathbf{u}^2 \circ \Pi^1)\Big]_{s^2} = 0.$$

We use the Newton law (5) and the condition (6) to obtain:

$$\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + \frac{1}{2}\Big[\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\Big]_{s^1} + \frac{1}{2}\Big[J^1\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\Big]_{s^1} = 0. \tag{17}$$

1. If $\|\dfrac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| < s^1$ and $\|J^1\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| < s^1$:
   $\dfrac{[\![\mathbf{u}]\!]_t^1}{2}(\dfrac{1}{\gamma^1} + \dfrac{J^1}{\gamma^2}) = 0$; so $[\![\mathbf{u}]\!]_t^1 = 0$. In this case we obtain: $\boldsymbol{\sigma}_t^1(\mathbf{u}^1) = \Big[\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\Big]_{s^1}$,
   and so: $\|\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| < s^1$.

2. If $\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| \geq s^1$ and $\|J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| \geq s^1$:

$$\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + \frac{s^1}{2} \frac{\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} + \frac{s^1}{2} \frac{J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} = 0. \tag{18}$$

The equation (18) shows that $\boldsymbol{\sigma}_t^1(\mathbf{u}^1)$ and $[\![\mathbf{u}]\!]_t^1$ are collinear.

$$\text{So:} \quad \begin{cases} \dfrac{\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} = \dfrac{J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|}, \\ \text{or} \\ \dfrac{\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} = -\dfrac{J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} (*), \end{cases}$$

$$\text{and we obtain, from (18) :} \begin{cases} \boldsymbol{\sigma}_t^1(\mathbf{u}^1) = -s^1 \frac{\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} = -\frac{1}{\gamma^1} [[\![\mathbf{u}]\!]_t^1 - \gamma^1 \boldsymbol{\sigma}_t^1(\mathbf{u}^1)]_{\gamma^1 s^1}, \\ \text{and this is equivalent to (4).} \\ \text{or} \\ \boldsymbol{\sigma}_t^1(\mathbf{u}^1) = 0 \text{ which is impossible in } (*). \end{cases}$$

3. If now $\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| < s^1$ and $\|J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| \geq s^1$:

$$\boldsymbol{\sigma}_t^1(\mathbf{u}^1) + \frac{1}{2}\big(\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\big) + \frac{s^1}{2} \frac{J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|J^1 \frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|} = 0.$$

Projecting on $\frac{\boldsymbol{\sigma}_t^1(\mathbf{u}^1)}{\|\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|}$ and laying $a = \|\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|$; $b = \dfrac{\boldsymbol{\sigma}_t^1(\mathbf{u}^1) \cdot [\![\mathbf{u}]\!]_t^1}{\gamma^1 \|\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|}$,

$$\text{we get:} \quad \begin{cases} |b - a| < s^1 \text{ and } |bJ^1 \frac{\gamma^1}{\gamma^2} - a| \geq s^1 \\ \text{and} \\ (b + a) + \epsilon s^1 = 0 \text{ ; where } \epsilon = \text{sign}(bJ^1 \frac{\gamma^1}{\gamma^2} - a) = \pm 1. \end{cases}$$

$$\text{Let } \epsilon = +1; \text{ so, } a = -b - s^1 \text{ and we obtain:} \begin{cases} b - a = 2b + s^1 \text{ and } |b - a| < s^1 \\ \text{and} \\ bJ^1 \frac{\gamma^1}{\gamma^2} - a = (J^1 \frac{\gamma^1}{\gamma^2} + 1)b + s^1 \text{ and } bJ^1 \frac{\gamma^1}{\gamma^2} - a \geq s^1. \end{cases}$$

$$\text{So:} \quad \begin{cases} -s^1 < b < 0 \\ \text{and} \\ (J^1 \frac{\gamma^1}{\gamma^2} + 1)b \geq 0 \end{cases} \quad \text{which is absurd.}$$

$$\text{Let } \epsilon = -1; \text{ so } a = -b + s^1 \text{ and we obtain:} \begin{cases} b - a = 2b - s^1 \text{ and } |b - a| < s^1 \\ \text{and} \\ bJ^1 \frac{\gamma^1}{\gamma^2} - a = (J^1 \frac{\gamma^1}{\gamma^2} + 1)b - s^1 \text{ and } bJ^1 \frac{\gamma^1}{\gamma^2} - a \leq -s^1, \end{cases}$$

$$\text{so:} \quad \begin{cases} 0 < b < s^1 \\ \text{and} \\ (J^1 \frac{\gamma^1}{\gamma^2} + 1)b \leq 0 \end{cases} \quad \text{, which is absurd.}$$

4. If $\|\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^1} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| \geq s^1$ and $\|J^1\frac{[\![\mathbf{u}]\!]_t^1}{\gamma^2} - \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\| < s^1$:

   We argue in the same way laying $a = \|\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|$; $b = J^1\dfrac{\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot[\![\mathbf{u}]\!]_t^1}{\gamma^2\|\boldsymbol{\sigma}_t^1(\mathbf{u}^1)\|}$.

Thus, we establish the friction condition (4) for i=1. In the same way, when supposing $\mathbf{v}^1 = 0$, we get (2a)-(2b)-(2c) and (4) for i=2. □

1.5 Discretization of the variational formulation

Let $\mathcal{T}_h^i$ a family of triangulations of the domain Ω^i supposed regular and conformal to the subdivisions of the boundaries into Γ_D^i, Γ_N^i and Γ_C^i. We introduce

$$\mathbf{V}_h = (\mathbf{V}_h^1 \times \mathbf{V}_h^2), \text{ with } \mathbf{V}_h^i = \left\{\mathbf{v}_h^i \in \mathscr{C}^0(\overline{\Omega^i}) : \mathbf{v}_{h|T}^i \in (\mathbb{P}_k(T))^d, \forall T \in \mathcal{T}_h^i, \mathbf{v}_h^i = 0 \text{ on } \Gamma_D^i\right\},$$

the family of finite dimensional vector spaces indexed by h and coming from $\mathcal{T}_h^i$.
We consider in what follows that γ^i is a positive piecewise constant function on the contact interface Γ_C^i which satisfies

$$\gamma^i_{|K^i\cap\Gamma_C^i} = \gamma_0 h_{K^i},$$

for every $K^i \in \mathcal{T}_h^i$ that has a non-empty intersection of dimension $d-1$ with Γ_C^i, and where γ_0 is a positive given constant. Note that the value of γ^i on element intersections has no influence. This allows to define a discrete counterpart of (13). Let us introduce for this purpose, with the same notation, the discrete linear operators:

$$\begin{aligned}
P_{n,\gamma^i}^i(\mathbf{u}_h) &= [\![u_h]\!]_n - g_n^i - \gamma^i\sigma_n^i(\mathbf{u}_h^i), & \mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h) &= [\![\mathbf{u}_h]\!]_t^i - \gamma^i\boldsymbol{\sigma}_t^i(\mathbf{u}_h^i),\\
P_{n,\theta\gamma^i}^i(\mathbf{v}_h) &= [\![v_h]\!]_n^i - \theta\gamma^i\sigma_n^i(\mathbf{v}_h^i), & \mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h) &= [\![\mathbf{v}_h]\!]_t^i - \theta\gamma^i\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i).
\end{aligned} \tag{19}$$

Then the unbiased formulation of the two bodies contact in the discrete setting reads:
(20)

$$\left\{\begin{aligned}
&\text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ such that, } \forall \mathbf{v}_h \in \mathbf{V}_h,\\
&A_\theta(\mathbf{u}_h,\mathbf{v}_h) + \frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma^1}P_{n,\theta\gamma^1}^1(\mathbf{v}_h)[P_{n,\gamma^1}^1(\mathbf{u}_h)]_+\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2}\frac{1}{\gamma^2}P_{n,\theta\gamma^2}^2(\mathbf{v}_h)[P_{n,\gamma^2}^2(\mathbf{u}_h)]_+\mathrm{d}\Gamma\\
&+\frac{1}{2}\int_{\Gamma_C^1}\frac{1}{\gamma^1}\mathbf{P}_{t,\theta\gamma^1}^1(\mathbf{v}_h)\cdot[\mathbf{P}_{t,\gamma^1}^1(\mathbf{u}_h)]_{\gamma^1 s^1}\mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2}\frac{1}{\gamma^2}\mathbf{P}_{t,\theta\gamma^2}^2(\mathbf{v}_h)\cdot[\mathbf{P}_{t,\gamma^2}^2(\mathbf{u}_h)]_{\gamma^2 s^2}\mathrm{d}\Gamma = L(\mathbf{v}_h).
\end{aligned}\right.$$

2 Mathematical analysis of the method

A major difference between Nitsche's method and classical penalty methods is the property of consistency demonstrated in 2.1. Using the same arguments as in [5] we prove the well-posedness and the optimal convergence of (20) when the mesh size h vanishes.

2.1 Consistency

Like Nitsche's method for unilateral contact problems [5], our Nitsche-based formulation (20) is consistent:

Lemma 2.1. *Suppose that the solution* $\mathbf{u}$ *of (1)-(5) lies in* $(H^{\frac{3}{2}+\nu}(\Omega^1))^d \times (H^{\frac{3}{2}+\nu}(\Omega^2))^d$ *with* $\nu > 0$*, then* $\mathbf{u}$ *is also solution of :*

(21)

$$A_\theta(\mathbf{u}, \mathbf{v}_h) + \frac{1}{2}\int_{\Gamma_C^1} \frac{1}{\gamma^1} P^1_{n,\theta\gamma^1}(\mathbf{v}_h)[P^1_{n,\gamma^1}(\mathbf{u})]_+ \mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2} P^2_{n,\theta\gamma^2}(\mathbf{v}_h)[P^2_{n,\gamma^2}(\mathbf{u})]_+ \mathrm{d}\Gamma$$
$$+ \frac{1}{2}\int_{\Gamma_C^1} \frac{1}{\gamma^1} \mathbf{P}^1_{t,\theta\gamma^1}(\mathbf{v}_h)\cdot[\mathbf{P}^1_{t,\gamma^1}(\mathbf{u})]_{\gamma^1 s^1} \mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2} \mathbf{P}^2_{t\theta\gamma^2}(\mathbf{v}_h)\cdot[\mathbf{P}^2_{t,\gamma^2}(\mathbf{u})]_{\gamma^2 s^2} \mathrm{d}\Gamma = L(\mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Proof. Let $\mathbf{u}$ be a solution of (1)–(5) and set $\mathbf{v}_h \in \mathbf{V}_h$. Since $\mathbf{u}^i \in (H^{\frac{3}{2}+\nu}(\Omega^i))^d$, we have $\sigma_n^i(\mathbf{u}^i) \in (H^\nu(\Gamma_C^i))^d$ and $P_{n\gamma^i}$ and $\mathbf{P}_{t\gamma^i}$ are well-defined and belong to $Ł^2(\Gamma_C^i)$.
With equations (1)–(4) and integration by parts, it holds:

$$\mathbf{a}(\mathbf{u}, \mathbf{v}_h) - \int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1) v^1_{hn} \mathrm{d}\Gamma - \int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2) v^2_{hn} \mathrm{d}\Gamma - \int_{\Gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot \mathbf{v}^1_{ht} \mathrm{d}\Gamma - \int_{\Gamma_C^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot \mathbf{v}^2_{ht} \mathrm{d}\Gamma = L(\mathbf{v}_h).$$

We use now (5) to write:

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) - \frac{1}{2}\int_{\Gamma_C^1} \sigma_n^1(\mathbf{u}^1)(v^1_{hn} + v^2_{hn}\circ\Pi^1) \mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2} \sigma_n^2(\mathbf{u}^2)(v^2_{hn} + v^1_{hn}\circ\Pi^2) \mathrm{d}\Gamma$$
$$- \frac{1}{2}\int_{\Gamma_C^1} \boldsymbol{\sigma}_t^1(\mathbf{u}^1)\cdot(\mathbf{v}^1_{ht} - \mathbf{v}^2_{ht}\circ\Pi^1) \mathrm{d}\Gamma - \frac{1}{2}\int_{\Gamma_C^2} \boldsymbol{\sigma}_t^2(\mathbf{u}^2)\cdot(\mathbf{v}^2_{ht} - \mathbf{v}^1_{ht}\circ\Pi^2) \mathrm{d}\Gamma = L(\mathbf{v}_h).$$

With the writings: For any $\theta \in \mathbb{R}$,

$$\begin{cases} v^1_{hn} + v^2_{hn}\circ\Pi^1 = (v^1_{hn} + v^2_{hn}\circ\Pi^1 - \theta\gamma^1\sigma_n^1(\mathbf{v}_h^1)) + \theta\gamma^1\sigma_n^1(\mathbf{v}_h^1) \\ v^2_{hn} + v^1_{hn}\circ\Pi^2 = (v^2_{hn} + v^1_{hn}\circ\Pi^2 - \theta\gamma^2\sigma_n^2(\mathbf{v}_h^2)) + \theta\gamma^2\sigma_n^2(\mathbf{v}_h^2) \end{cases}$$

$$\begin{cases} \mathbf{v}^1_{th} - \mathbf{v}^2_{ht}\circ\Pi^1 = (\mathbf{v}^1_{ht} - \mathbf{v}^2_{ht}\circ\Pi^1 - \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}_h^1)) + \theta\gamma^1\boldsymbol{\sigma}_t^1(\mathbf{v}_h^1) \\ \mathbf{v}^2_{th} - \mathbf{v}^1_{ht}\circ\Pi^2 = (\mathbf{v}^2_{ht} - \mathbf{v}^1_{ht}\circ\Pi^2 - \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}_h^2)) + \theta\gamma^2\boldsymbol{\sigma}_t^2(\mathbf{v}_h^2), \end{cases}$$

the writing (7) of the contact conditions and the notations (12), we obtain (21). □

2.2 Well-posedness

To prove well-posedness of our formulation, we first need the following classical discrete trace inequality.

Lemma 2.2. *There exists* $C > 0$*, independent of the parameter* γ_0 *and of the mesh size* h*, such that:*

$$\|{\gamma^i}^{\frac{1}{2}}\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i)\|^2_{0,\Gamma_c^i} + \|{\gamma^i}^{\frac{1}{2}}\sigma_n^i(\mathbf{v}_h^i)\|^2_{0,\Gamma_c^i} \leq C\gamma_0\|\mathbf{v}_h^i\|^2_{1,\Omega^i}\ , \tag{22}$$

for all $\mathbf{v}_h^i \in \mathbf{V}_h^i$.

Proof. The inequality (22) is obtained using a scaling argument as in [4] Lemma 3.2 . □

We then show in Theorem 2.3 that the problem (20) is well-posed using an argument in ([3]) for M-type and pseudo-monotone operators. In the proof of the well-posedness, two cases are discused: $\theta = 1$ and $\theta \neq 1$.

Theorem 2.3. *Suppose that $\gamma_0 > 0$ is sufficiently small or $\theta = -1$, then Problem (20) admits one unique solution $\mathbf{u}_h$ in $\mathbf{V}_h$. When $\theta = -1$ we do not need the assumption of smallness of γ_0.*

Proof. Using the Riesz representation theorem, we define a (non-linear) operator $\mathbf{B} : \mathbf{V}_h \to \mathbf{V}_h$, by means of the formula:

$$
\begin{aligned}
(\mathbf{B}\mathbf{u}_h, \mathbf{v}_h)_1 &= A_\theta(\mathbf{u}_h, \mathbf{v}_h) + \frac{1}{2}\int_{\Gamma_C^1} \frac{1}{\gamma^1} P^1_{n,\theta\gamma^1}(\mathbf{v}_h)[P^1_{n,\gamma^1}(\mathbf{u}_h)]_+ \mathrm{d}\Gamma \\
&+ \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2} P^2_{n,\theta\gamma^2}(\mathbf{v}_h)[P^2_{n,\gamma^2}(\mathbf{u}_h)]_+ \mathrm{d}\Gamma + \frac{1}{2}\int_{\Gamma_C^1} \frac{1}{\gamma^1} \mathbf{P}^1_{t,\theta\gamma^1}(\mathbf{v}_h)\cdot[\mathbf{P}^1_{t,\gamma^1}(\mathbf{u}_h)]_{\gamma^1 s^1} \mathrm{d}\Gamma \\
&+ \frac{1}{2}\int_{\Gamma_C^2} \frac{1}{\gamma^2} \mathbf{P}^2_{t,\theta\gamma^2}(\mathbf{v}_h)\cdot[\mathbf{P}^2_{t,\gamma^2}(\mathbf{u}_h)]_{\gamma^2 s^2} \mathrm{d}\Gamma,
\end{aligned}
$$

for all $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$, and where $(.,.)_1$ stands for the scalar product in $\mathbf{V}$.
Note that Problem (20) is well-posed if and only if $\mathbf{B}_h$ is a one-to-one operator. Let $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$. Using the expression $P^n_{\theta\gamma}(.) = P^n_\gamma(.) + (1-\theta)\sigma_n(.)$ (and same for $\mathbf{P}^t_{\theta\gamma}$) we have:

$$
\begin{aligned}
&(\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 = \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) \\
&+\sum_{i=1}^{2}\Big(-\frac{\theta}{2}\|{\gamma^i}^{\frac{1}{2}}\boldsymbol{\sigma}(\mathbf{v}_h^i - \mathbf{w}_h^i)\mathbf{n}\|^2_{0,\Gamma_C^i} \\
&+\frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i} P^i_{n,\gamma^i}(\mathbf{v}_h - \mathbf{w}_h)\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\mathrm{d}\Gamma \\
&+\frac{(1-\theta)}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}\gamma^i\sigma^i_n(\mathbf{v}_h^i - \mathbf{w}_h^i)\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\mathrm{d}\Gamma \\
&+\frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i} \mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h - \mathbf{w}_h).\big([\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\mathrm{d}\Gamma \\
&+\frac{(1-\theta)}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}\gamma^i\boldsymbol{\sigma}^i_t(\mathbf{v}_h^i - \mathbf{w}_h^i).\big([\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h^i)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\Big)
\end{aligned}
$$

We use Cauchy-Schwarz inequality and the proprieties (9) and (10) to get:

$$
\begin{aligned}
&(\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) + \sum_{i=1}^{2}\Big(-\frac{\theta}{2}\|{\gamma^i}^{\frac{1}{2}}\boldsymbol{\sigma}(\mathbf{v}_h^i - \mathbf{w}_h^i)\mathbf{n}\|^2_{0,\Gamma_C^i} \\
&+\frac{1}{2}\|{\gamma^i}^{-\frac{1}{2}}\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\|^2_{0,\Gamma_C^i} + \frac{1}{2}\|{\gamma^i}^{-\frac{1}{2}}\big([\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\|^2_{0,\Gamma_C^i} \\
&-\frac{|1-\theta|}{2}\|{\gamma^i}^{-\frac{1}{2}}\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\|_{0,\Gamma_C^i}\|{\gamma^i}^{\frac{1}{2}}\sigma^i_n(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i} \\
&-\frac{|1-\theta|}{2}\|{\gamma^i}^{-\frac{1}{2}}\big([\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\|_{0,\Gamma_C^i}\|{\gamma^i}^{\frac{1}{2}}\boldsymbol{\sigma}^i_t(\mathbf{v}_h^i - \mathbf{w}_h^i)\|_{0,\Gamma_C^i}\Big)
\end{aligned}
$$

If $\theta = 1$, we use the coercivity of $a(\cdot,\cdot)$ and the property (22) to get:

$$\begin{aligned}
&(\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) - \sum_{i=1}^{2} \frac{1}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\mathbf{n}^i\|^2_{0,\Gamma_C^i} \\
&\geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) - \sum_{i=1}^{2} \frac{1}{2}\Big(\|\gamma^{i\frac{1}{2}} \sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i} + \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i}\Big) \\
&\geq C\|\mathbf{v}_h - \mathbf{w}_h\|_1^2
\end{aligned}$$

when γ_0 is sufficiently small.
We suppose now that $\theta \neq 1$; let $\beta > 0$. Applying Young inequality yields:

$$\begin{aligned}
&(\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 \geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) + \sum_{i=1}^{2} \Big(-\frac{\theta}{2} \|\gamma^{i\frac{1}{2}} \boldsymbol{\sigma}^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\mathbf{n}^i\|^2_{0,\Gamma_C^i} \\
&+\frac{1}{2}\|\gamma^{i-\frac{1}{2}}\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\|^2_{0,\Gamma_C^i} + \frac{1}{2}\|\gamma^{i-\frac{1}{2}}\big([\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\|^2_{0,\Gamma_C^i} \\
&-\frac{|1-\theta|}{4\beta}\|\gamma^{i-\frac{1}{2}}\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\|^2_{0,\Gamma_C^i} - \frac{|1-\theta|\beta}{4}\|\gamma^{i\frac{1}{2}}\sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i} \\
&-\frac{|1-\theta|}{4\beta}\|\gamma^{i-\frac{1}{2}}\big(\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\|^2_{0,\Gamma_C^i} - \frac{|1-\theta|\beta}{4}\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i}\Big) \\
&= \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) + \sum_{i=1}^{2}\Big(-\frac{1}{2}\Big(\theta + \frac{|1-\theta|\beta}{2}\Big)\Big(\|\gamma^{i\frac{1}{2}}\sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i} \\
&+\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i}\Big) + \frac{1}{2}\Big(1 - \frac{|1-\theta|}{2\beta}\Big)\Big(\|\gamma^{i-\frac{1}{2}}\big([P^i_{n,\gamma^i}(\mathbf{v}_h)]_+ - [P^i_{n,\gamma^i}(\mathbf{w}_h)]_+\big)\|^2_{0,\Gamma_C^i} \\
&+\|\gamma^{i-\frac{1}{2}}\big([\mathbf{P}^i_{t,\gamma^i}(\mathbf{v}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t,\gamma^i}(\mathbf{w}_h)]_{\gamma^i s^i}\big)\|^2_{0,\Gamma_C^i}\Big)\Big)
\end{aligned}$$

Choosing $\beta = \dfrac{|1-\theta|}{2}$ and γ_0 sufficiently small we get:

$$\begin{aligned}
(\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 &\geq \mathbf{a}(\mathbf{v}_h - \mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h) - \frac{(1+\theta)^2}{8}\sum_{i=1}^{2}\Big(\|\gamma^{i\frac{1}{2}}\sigma_n^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i} \\
&\qquad + \|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i - \mathbf{w}_h^i)\|^2_{0,\Gamma_C^i}\Big). \\
(\mathbf{B}\mathbf{v}_h - \mathbf{B}\mathbf{w}_h, \mathbf{v}_h - \mathbf{w}_h)_1 &\geq C\|\mathbf{v} - \mathbf{w}\|_1^2
\end{aligned}$$

Note that, when $\theta = -1$ we do not need the assumption of smallness of γ_0.
Let us show, now, that $\mathbf{B}$ is hemicontinuous. Since $\mathbf{V}^h$ is a vector space, it is sufficient to show that:

$$\begin{aligned}
\varphi : \quad [0,1] &\to \mathbb{R} \\
t &\mapsto (\mathbf{B}(\mathbf{v}_h - t\mathbf{w}_h), \mathbf{w}_h)_1
\end{aligned}$$

is a continuous real function for all $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$. Let $t, s \in [0,1]$, we compute:

$$
\begin{aligned}
|\varphi(t)-\varphi(s)| &= \Big|(\mathbf{B}(\mathbf{v}_h - t\mathbf{w}_h) - \mathbf{B}(\mathbf{v}_h - s\mathbf{w}_h), \mathbf{w}_h)_1\Big| \\
&= \Big| A_\theta((s-t)\mathbf{w}_h, \mathbf{w}_h) + \sum_{i=1}^{2}\Big(\frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i} P^i_{n,\theta\gamma^i}(\mathbf{w}_h)\big([P^i_{n\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h)]_+ - [P^i_{n\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)]_+\big)\mathrm{d}\Gamma \\
&+ \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}\mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{w}_h)\big([\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\Big)\Big| \\
&\le |s-t| A_\theta(\mathbf{w}_h, \mathbf{w}_h) + \sum_{i=1}^{2}\Big(\frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i} |P^i_{n,\theta\gamma^i}(\mathbf{w}_h)|\Big|[P^i_{n\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h)]_+ - [P^i_{n\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)]_+\Big|\mathrm{d}\Gamma \\
&+ \frac{1}{2}\int_{\Gamma_C^i} \frac{1}{\gamma^i}\|\mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{w}_h)\|\Big\|[\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)]_{\gamma^i s^i}\Big\|\mathrm{d}\Gamma\Big)
\end{aligned}
$$

We use the bounds $|[a]_+ - [b]_+| \le |a-b|$ for all $a, b \in \mathbb{R}$ and $\big\|[\mathbf{a}]_{\gamma^i g^i} - [\mathbf{b}]_{\gamma^i g^i}\big\| \le \|\mathbf{a}-\mathbf{b}\|$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{d-1}$ to deduce that:

$$
\begin{aligned}
&\int_{\Gamma_C^i} \frac{1}{\gamma^i}|P^i_{n,\theta\gamma^i}(\mathbf{w}_h)|\Big|[P^i_{n\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h)]_+ - [P^i_{n\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)]_+\Big|\mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_C^i} \frac{1}{\gamma^i}\|\mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{w}_h)\|\Big\|[\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h)]_{\gamma^i s^i} - [\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)]_{\gamma^i s^i}\Big\|\mathrm{d}\Gamma \\
&\le \int_{\Gamma_C^i} \frac{1}{\gamma^i}|P^i_{n,\theta\gamma^i}(\mathbf{w}_h)|\Big|P^i_{n\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h) - P^i_{n\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)\Big|\mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_C^i} \frac{1}{\gamma^i}\|\mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{w}_h)\|\Big\|\mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - t\mathbf{w}_h) - \mathbf{P}^i_{t\gamma^i}(\mathbf{v}_h - s\mathbf{w}_h)\Big\|\mathrm{d}\Gamma \\
&\le |s-t|\Big(\int_{\Gamma_C^i} \frac{1}{\gamma^i}|P^i_{n,\theta\gamma^i}(\mathbf{w}_h)||P^i_{n\gamma^i}(\mathbf{w}_h)|\mathrm{d}\Gamma + \int_{\Gamma_C^i} \frac{1}{\gamma^i}\|\mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{w}_h)\|\|\mathbf{P}^i_{t\gamma^i}(\mathbf{w}_h)\|\mathrm{d}\Gamma\Big)
\end{aligned}
$$

It results that:

$$
\begin{aligned}
|\varphi(t)-\varphi(s)| \quad \le \quad & |s-t|\Big(A_\theta(\mathbf{w}_h, \mathbf{w}_h) + \sum_{i=1}^{2}\Big(\int_{\Gamma_C^i} \frac{1}{2\gamma^i}|P^i_{n,\theta\gamma^i}(\mathbf{w}_h)||P^i_{n\gamma^i}(\mathbf{w}_h)|\mathrm{d}\Gamma \\
&+ \int_{\Gamma_C^i} \frac{1}{2\gamma^i}\|\mathbf{P}^i_{t,\theta\gamma^i}(\mathbf{w}_h)\|\|\mathbf{P}^i_{t\gamma^i}(\mathbf{w}_h)\|\mathrm{d}\Gamma\Big)\Big).
\end{aligned}
$$

Which means that φ is Lipschitz, so that $\mathbf{B}$ is hemicontinuous. We finally apply the Corollary 15 (p.126) of [3] to conclude that $\mathbf{B}$ is a one to one operator. □

2.3 A priori error analysis

Our Nitsche-based method (20) converges in a optimal way as the mesh parameter h vanishes. This is proved in the Theorem 2.5, where we provide an estimate of the displacement error in H^1-norm and of the contact error in $L^2(\Gamma_C^i)$-norm. We establish, first, the following abstract error estimate.

Theorem 2.4. *Suppose that* $\mathbf{u}$ *is a solution to (1-5) and belongs to* $(H^{\frac{3}{2}+\nu}(\Omega^1))^d \times (H^{\frac{3}{2}+\nu}(\Omega^2))^d$ *with* $\nu > 0$.

1. We suppose γ_0 sufficiently small. The solution $\mathbf{u}_h$ to the discrete problem (20) satisfies the following error estimate:
(23)
$$\sum_{i=1}^{2}\Big(\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}^2+\frac{1}{2}\|{\gamma^i}^{\frac{1}{2}}\big(\sigma_n^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[P_{n,\gamma}^i(\mathbf{u}_h)]_+\big)\|_{0,\Gamma_C^i}^2+\frac{1}{2}\|{\gamma^i}^{\frac{1}{2}}\big(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}\big)\|_{0,\Gamma_C^i}^2\Big)$$
$$\leq C\inf_{\mathbf{v}_h\in\mathbf{V}_h}\Big(\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{v}_h^i\|_{1,\Omega^i}^2+\frac{1}{2}\|{\gamma^i}^{-\frac{1}{2}}(\mathbf{u}^i-\mathbf{v}_h^i)\|_{0,\Gamma_C^i}^2+\frac{1}{2}\|{\gamma^i}^{\frac{1}{2}}\boldsymbol{\sigma}(\mathbf{u}^i-\mathbf{v}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2\Big),$$
where $C>0$ is a constant independent of h, $\mathbf{u}$ and γ_0.

2. If $\theta=-1$, for all $\gamma_0>0$, the solution $\mathbf{u}_h$ to the problem (20) satisfies the estimate (23) with $C>0$ a constant independent of h and $\mathbf{u}$, but eventually dependent of γ_0.

Proof. Let $\mathbf{v}_h\in\mathbf{V}_h$, using the coercivity and the continuity of the form $a(\cdot,\cdot)$ as well as Young's inequality, we obtain:

$$\begin{aligned}
\alpha\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}^2 &\leq a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h)\\
&= a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{v}_h)+a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)\\
&\leq C\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}\|\mathbf{u}^i-\mathbf{v}_h^i\|_{1,\Omega^i}+a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)\\
&\leq \frac{\alpha}{2}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}^2+\frac{C^2}{2\alpha}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{v}_h^i\|_{1,\Omega^i}^2\\
&\quad +a(\mathbf{u},\mathbf{v}_h-\mathbf{u}_h)-a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h).
\end{aligned}$$

Therefore, we get:

$$\frac{\alpha}{2}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}^2\leq\frac{C^2}{2\alpha}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{v}_h^i\|_{1,\Omega^i}^2+a(\mathbf{u},\mathbf{v}_h-\mathbf{u}_h)-a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h).$$

Since $\mathbf{u}$ solves (1-5) and $\mathbf{u}_h$ solves (20), using the Lemma 2.1 yields:

$$\begin{aligned}
\frac{\alpha}{2}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}^2 \leq\ & \frac{C^2}{2\alpha}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{v}_h^i\|_{1,\Omega^i}^2+\sum_{i=1}^{2}\Big(-\frac{\theta}{2}\int_{\Gamma_C^i}\gamma^i\boldsymbol{\sigma}^i(\mathbf{u}_h^i-\mathbf{u}^i)\mathbf{n}^i\cdot\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\mathrm{d}\Gamma\\
&+\frac{1}{2}\int_{\Gamma_C^i}\frac{1}{\gamma^i}\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h-\mathbf{u}_h)\cdot\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}-[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\\
&+\frac{1}{2}\int_{\Gamma_C^i}\frac{1}{\gamma^i}P_{n,\theta\gamma^i}^i(\mathbf{v}_h-\mathbf{u}_h)\big([P_{n,\gamma^i}^i(\mathbf{u}_h)]_+-[P_{n,\gamma^i}^i(\mathbf{u})]_+\big)\mathrm{d}\Gamma\Big).
\end{aligned} \tag{24}$$

Let $\beta_1>0$. The first integral term in (24) is bounded, using Cauchy-Schwarz and Young's inequalities, as follows:

(25)

$$
\begin{aligned}
&-\frac{\theta}{2}\int_{\Gamma_C^i}\gamma^i\boldsymbol{\sigma}^i(\mathbf{u}_h^i-\mathbf{u}^i)\mathbf{n}^i\cdot\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\mathrm{d}\Gamma\\
&=\frac{\theta}{2}\int_{\Gamma_C^i}\gamma^i\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\cdot\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\mathrm{d}\Gamma-\frac{\theta}{2}\int_{\Gamma_C^i}\gamma^i\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}^i)\mathbf{n}^i\cdot\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\mathrm{d}\Gamma\\
&\leq\frac{\theta}{2}\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2+\frac{|\theta|}{2}\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}^i)\mathbf{n}^i\|_{0,\Gamma_C^i}\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}\\
&\leq\frac{\beta_1\theta^2}{4}\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2+\frac{1}{2}\big(\theta+\frac{1}{2\beta_1}\big)\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2.
\end{aligned}
$$

For the second integral term in (24), we can write:

$$
\begin{aligned}
&\int_{\Gamma_C^i}\frac{1}{\gamma^i}\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h-\mathbf{u}_h)\cdot\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}-[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\\
&=\int_{\Gamma_C^i}\frac{1}{\gamma^i}\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h-\mathbf{u})\cdot\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}-[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\\
&+\int_{\Gamma_C^i}\frac{1}{\gamma^i}\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}-\mathbf{u}_h)\cdot\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}-[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\\
&+\int_{\Gamma_C^i}(1-\theta)\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\cdot\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}-[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\mathrm{d}\Gamma.
\end{aligned}
$$

Using the bound (10) and applying two times Cauchy-Schwarz and Young's inequalities, we obtain for $\beta_2>0$ and $\beta_3>0$:

(26)

$$
\begin{aligned}
&\int_{\Gamma_C^i}\frac{1}{\gamma^i}\mathbf{P}_{t,\theta\gamma^i}^i(\mathbf{v}_h-\mathbf{u}_h)\cdot\big([\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}-[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u})]_{\gamma^i s^i}\big)\mathrm{d}\Gamma\\
&\leq\frac{1}{2\beta_2}\Big\|\gamma^{i\frac{1}{2}}\Big(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}\Big)\Big\|_{0,\Gamma_C^i}^2+\frac{\beta_2}{2}\|\gamma^{i-\frac{1}{2}}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h-\mathbf{u})]_{\gamma^i s^i}\|_{0,\Gamma_C^i}^2\\
&-\Big\|\gamma^{i\frac{1}{2}}\Big(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}\Big)\Big\|_{0,\Gamma_C^i}^2+\frac{|1-\theta|}{2\beta_3}\Big\|\gamma^{i\frac{1}{2}}\Big(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma^i s^i}\Big)\Big\|_{0,\Gamma_C^i}^2\\
&+\frac{|1-\theta|\beta_3}{2}\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}_t^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\|_{0,\Gamma_C^i}^2.
\end{aligned}
$$

In a similar way, we can upper bound the third integral term of (24).

Noting that:

$$
\begin{aligned}
&\|\gamma^{i-\frac{1}{2}}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{v}_h-\mathbf{u})]_{\gamma^i s^i}\|_{0,\Gamma_C^i}^2\\
&+\|\gamma^{i-\frac{1}{2}}[P_{n,\gamma^i}^i(\mathbf{v}_h-\mathbf{u})]_+\|_{0,\Gamma_C^i}^2\leq 2\|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i-\mathbf{v}_h^i)\|_{0,\Gamma_C^i}^2+2\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{u}^i-\mathbf{v}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2,
\end{aligned}
\tag{27}
$$

and using estimates (25) and (26) in (24), we obtain:

(28)

$$
\begin{aligned}
&\frac{\alpha}{2}\sum_{i=1}^2\|\mathbf{u}^i-\mathbf{u}_h^i\|_{1,\Omega^i}^2\leq\frac{C^2}{2\alpha}\sum_{i=1}^2\|\mathbf{u}^i-\mathbf{v}_h^i\|_{1,\Omega^i}^2\\
&+\frac{1}{2}\sum_{i=1}^2\Big(\big(\frac{\beta_1\theta^2}{2}+\beta_2\big)\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{u}^i-\mathbf{v}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2+\beta_2\|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i-\mathbf{v}_h^i)\|_{0,\Gamma_C^i}^2\\
&+\big(-1+\frac{1}{2\beta_2}+\frac{|1-\theta|}{2\beta_3}\big)\big(\|\gamma^{i\frac{1}{2}}(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}_{t,\gamma^i}^i(\mathbf{u}_h)]_{\gamma s^i})\|_{0,\Gamma_C^i}^2+\|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i)+\frac{1}{\gamma}[P_{n,\gamma^i}^i(\mathbf{u}_h)]_+)\|_{0,\Gamma_C^i}^2\big)\\
&+\big(\frac{1}{2\beta_1}+\theta+\frac{|1-\theta|\beta_3}{2}\big)\big(\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|_{0,\Gamma_C^i}^2\big)\Big).
\end{aligned}
$$

We use now the estimate (22) to get:

$$(29) \qquad \|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|^2_{0,\Gamma^i_C} \le C\gamma_0^{\frac{1}{2}}\|\mathbf{v}_h^i-\mathbf{u}_h^i\|^2_{1,\Omega^i} \le C\gamma_0^{\frac{1}{2}}(\|\mathbf{v}_h^i-\mathbf{u}^i\|^2_{1,\Omega^i}+\|\mathbf{u}_h^i-\mathbf{u}^i\|^2_{1,\Omega^i})$$

For a fixed $\theta \in \mathbb{R}$ we choose β_2 and β_3 large enough that:

$$-1+\frac{1}{2\beta_2}+\frac{|1-\theta|}{2\beta_3} < -\frac{1}{2}$$

Choosing γ_0 small enough in (29) and putting the estimate in (28), we establish the first statement of the theorem.

We consider now the case $\theta=-1$ in which (28) becomes:

$$\begin{aligned}
&\frac{\alpha}{2}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|^2_{1,\Omega^i} \le \frac{C^2}{2\alpha}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{v}_h^i\|^2_{1,\Omega^i}\\
&+\frac{1}{2}\sum_{i=1}^{2}\Big((\frac{\beta_1}{2}+\beta_2)\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{u}^i-\mathbf{v}_h^i)\mathbf{n}^i\|^2_{0,\Gamma^i_C}+\beta_2\|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i-\mathbf{v}_h^i)\|^2_{0,\Gamma^i_C}\\
&+\big(-1+\frac{1}{2\beta_2}+\frac{1}{\beta_3}\big)\big(\|\gamma^{i\frac{1}{2}}(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}^i_{t,\gamma^i}(\mathbf{u}_h)]_{\gamma^i s^i})\|^2_{0,\Gamma^i_C}+\|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[P^i_{n,\gamma^i}(\mathbf{u}_h)]_+)\|^2_{0,\Gamma^i_C}\big)\\
&+\big(\frac{1}{2\beta_1}-1+\beta_3\big)\big(\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|^2_{0,\Gamma^i_C}\big)\Big).
\end{aligned}$$

Let be given $\eta>0$. Set $\beta_1=\frac{1}{2\eta}$, $\beta_2=1+\frac{1}{\eta}$, $\beta_3=1+\eta$. And so we arrive at:

$$\begin{aligned}
&\frac{\alpha}{2}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{u}_h^i\|^2_{1,\Omega^i} \le \frac{C^2}{2\alpha}\sum_{i=1}^{2}\|\mathbf{u}^i-\mathbf{v}_h^i\|^2_{1,\Omega^i}\\
&+\frac{1}{2}\sum_{i=1}^{2}\Big((\frac{5}{4\eta}+1)\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{u}^i-\mathbf{v}_h^i)\mathbf{n}^i\|^2_{0,\Gamma^i_C}+\frac{1+\eta}{\eta}\|\gamma^{i-\frac{1}{2}}(\mathbf{u}^i-\mathbf{v}_h^i)\|^2_{0,\Gamma^i_C}\\
&-\frac{\eta}{2(1+\eta)}\big(\|\gamma^{i\frac{1}{2}}(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}^i_{t,\gamma^i}(\mathbf{u}_h)]_{\gamma^i s^i})\|^2_{0,\Gamma^i_C}+\|\gamma^{i\frac{1}{2}}(\sigma_n^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[P^i_{n,\gamma^i}(\mathbf{u}_h)]_+)\|^2_{0,\Gamma^i_C}\big)\\
&+2\eta\|\gamma^{i\frac{1}{2}}\boldsymbol{\sigma}^i(\mathbf{v}_h^i-\mathbf{u}_h^i)\mathbf{n}^i\|^2_{0,\Gamma^i_C}\Big)
\end{aligned}$$

Set $\eta=\dfrac{\alpha}{16C^2\gamma_0}$, where C is the constant in (29) to conclude the proof of the theorem. □

Theorem 2.5. *Suppose that* $\mathbf{u}=(\mathbf{u}^1,\mathbf{u}^2)$ *is a solution to problem (1-5) and belongs to* $(H^{\frac{3}{2}+\nu}(\Omega^1))^d\times(H^{\frac{3}{2}+\nu}(\Omega^2))^d$ *with* $0<\nu\le\dfrac{1}{2}$ *if* $k=1$ *and* $0<\nu\le 1$ *if* $k=2$ *(k is the degree of the finite element method). If* $\theta=-1$ *or* γ_0 *is sufficiently small, the solution* $\mathbf{u}_h$ *to the problem (20) satisfies the following estimate:*

(30)

$$\begin{aligned}
&\sum_{i=1}^{2}\Big(\|\mathbf{u}^i-\mathbf{u}_h^i\|^2_{1,\Omega^i}+\frac{1}{2}\|\gamma^{i\frac{1}{2}}\big(\sigma_n^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[P^i_{n,\gamma^i}(\mathbf{u}_h)]_+\big)\|^2_{0,\Gamma^i_C}+\frac{1}{2}\|\gamma^{i\frac{1}{2}}\big(\boldsymbol{\sigma}_t^i(\mathbf{u}^i)+\frac{1}{\gamma^i}[\mathbf{P}^i_{t,\gamma^i}(\mathbf{u}_h)]_{\gamma^i s^i}\big)\|^2_{0,\Gamma^i_C}\Big)\\
&\le Ch^{\frac{1}{2}+\nu}\sum_{i=1}^{2}\|\mathbf{u}^i\|^2_{\frac{3}{2}+\nu,\Omega^i}
\end{aligned}$$

where C is a constant independent of $\mathbf{u}$ *and* h.

Proof. To establish (30) we need to bound the right terms in estimate (23). We choose $\mathbf{v}_h^i = \mathcal{I}_h^i \mathbf{u}^i$ where $\mathcal{I}_h^i$ stands for the Lagrange interpolation operator mapping onto $\mathbf{V}_h^i$. The estimation of the Lagrange interpolation error in the H^1-norm on a domain is classical (see, e.g., [8], [2] and [9])

$$\|\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i\|_{1,\Omega^i} \leq C h^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i} \tag{31}$$

for $-\frac{1}{2} < \nu \leq k - \frac{1}{2}$.

Let E in Γ_C^i be an edge of triangle $K \in T_h^i$, we have:

$$\|\gamma^{i^{-\frac{1}{2}}}(\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i)\|_{0,E} \leq C h^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{1+\nu,E} \leq C h^{\frac{1}{2}} \|\mathbf{u}^i\|_{1+\nu,E}$$

A summation on all the edges E , with the trace theorem yields:

$$\|\gamma^{i^{-\frac{1}{2}}}(\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i)\|_{0,\Gamma_C^i} \leq C h_k^{\frac{1}{2}+\nu} \|\mathbf{u}^i\|_{1+\nu,\Gamma_C^i} \leq C h^{\frac{1}{2}} \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i} \tag{32}$$

From Appendix A of [7] (see also [11]), we get the following estimate:

$$\|\gamma^{i^{\frac{1}{2}}} \boldsymbol{\sigma}(\mathbf{u}^i - \mathcal{I}_h^i \mathbf{u}^i)\mathbf{n}^i\|_{0,\Gamma_C^i} \leq C h^{\frac{1}{2}} \|\mathbf{u}^i\|_{\frac{3}{2}+\nu,\Omega^i} \tag{33}$$

By inserting (31), (32) and (33) onto (23) we get (30). □

3 Numerical experiments

In this section, we test the Nitsche's unbiased method (20) for two/three-dimensional contact between two elastic bodies Ω^1 and Ω^2. The first body is a disk/sphere and the second is a rectangle/rectangular cuboid. This situation is not strictly a Hertz type contact problem because Ω^2 is bounded.
The tests are performed with P_1 and P_2 Lagrange finite elements. The finite element library Getfem++ is used. The discrete contact problem is solved by using a generalized Newton method. Further details on generalized Newton's method applied to contact problems can be found for instance in [21] and the references therein.
The accuracy of the method is discussed for the different cases with respect to the finite element used, the mesh size, and the value of the parameters θ and γ_0. We perform experiences with a frictionless contact to compare the results of the formulation with other formulations using Nitsche's method (given mainly in [10] and [7]). Moreover, we present the convergence curves for a frictional contact in figures 10 and 11.

The numerical tests in two dimensions (resp. three dimensions) are performed on a domain $\Omega =]-0.5, 0.5[^2$ (resp. $\Omega =]-0.5, 0.5[^3$) containing the two bodies Ω^1 and Ω^2 . The first body is a disk of radius 0.25 and center (0,0) (resp. a sphere of radius 0.25 and center (0,0,0)), and the second is rectangle $]-0.5, 0.5[\times]-0.5, -0.25[$ (resp. $\Omega^2 =]-0.5, 0.5[^2\times]-0.5, 0.25[$). The contact surface Γ_C^1 is the lower semicircle and Γ_C^2 is the top surface of Ω^2 (i.e. $\Gamma_C^1 = \{\mathbf{x} \in \partial\Omega^1; x_2 \leq 0\}$ and $\Gamma_C^2 = \{\mathbf{x} \in \partial\Omega^2; x_2 = -0.25\}$ (resp. $\Gamma_C^1 = \{\mathbf{x} \in \partial\Omega^1; x_3 \leq 0\}$ and $\Gamma_C^2 = \{\mathbf{x} \in \partial\Omega^2; x_3 = -0.25\}$)). A Dirichlet condition is prescribed on the bottom of the rectangle (resp. cuboid). Since no Dirichlet condition is applied on Ω^1 the problem is only semi-coercive. To overcome the non-definiteness coming from the free rigid motions, the horizontal displacement is prescribed to be zero on the two points of coordinates (0,0) and (0,0.1) (resp. (0,0,0) and (0,0,0.1)) which blocks the horizontal translation and the rigid rotation. The projector Π^1 is defined from Γ_C^1 to Γ_C^2 in the vertical direction. All remaining parts of the boundaries are considered traction free. For simplicity, we consider a dimensionless configuration with Lamé coefficients $\lambda = 1$ and $\mu = 1$ and a volume density of vertical force $f_v = -0.25$.

The expression of the exact solution being unknown, the convergence is studied with respect to a reference solution computed with a P_2 element on a very fine mesh for $\theta = -1$. (see Figures 2 and 3).

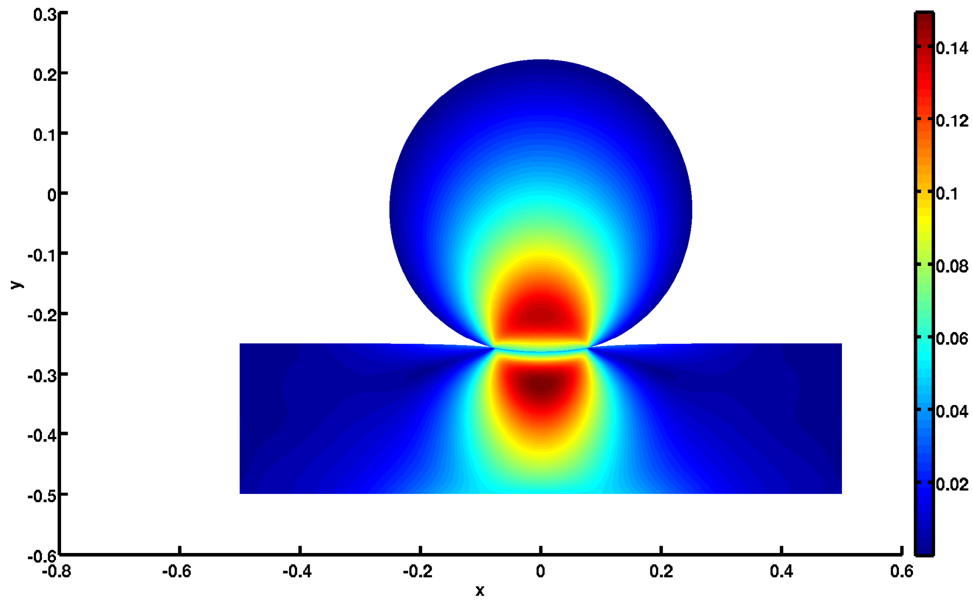


Figure 2: 2D Numerical reference solution with contour plot of Von Mises stress. $h = 1/400, \gamma_0 = 1/100$ and P_2 elements.

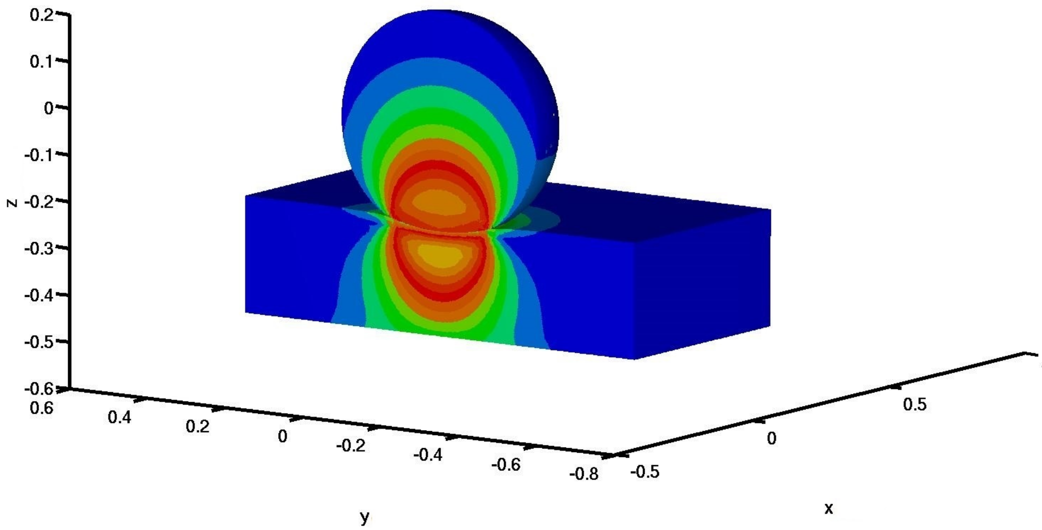


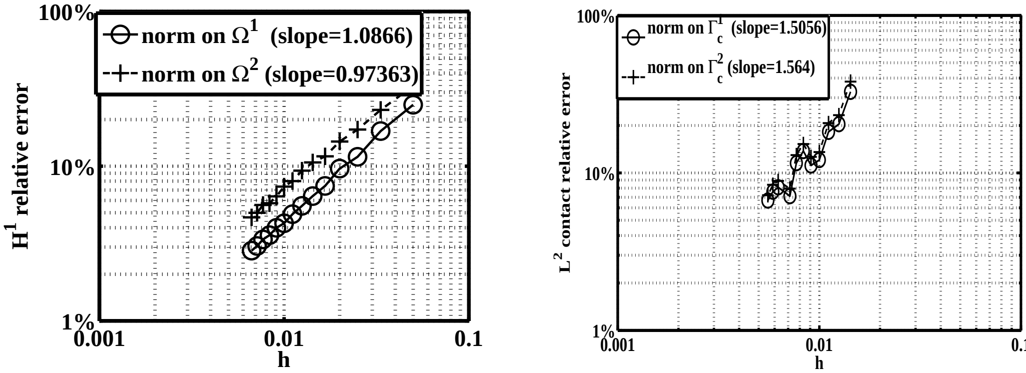
Figure 3: Cross-section of 3D numerical reference solution with contour plot of Von Mises stress. $h = 1/50, \gamma_0 = 1/100$ and P_2 elements.

3.1 Convergence in the two dimensional frictionless case

We perform a numerical convergence study on the three methods $\theta = 1$, $\theta = 0$ and $\theta = -1$ for a fixed parameter $\gamma_0 = \frac{1}{100}$ (chosen small in order to have the convergence for the three cases) and friction coefficients $s1 = s2 = 0$. In each case we plot the relative error in percentage in the H^1-norm of the displacement in the two bodies and the error of the L^2 norm of the Nitsche's contact condition on Γ_C^1 and Γ_C^2. The error of the Nitsche's contact condition is equal to:

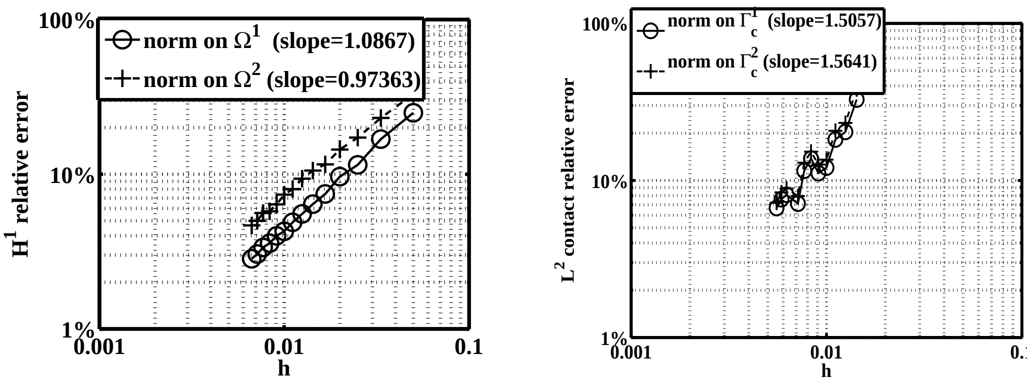
$$\frac{\|\gamma^{i\,\frac{1}{2}}\left(\sigma_n^i(\mathbf{u}_{ref}^{hi}) + \frac{1}{\gamma^i}[P_{n,\gamma}^i(\mathbf{u}_h)]_+\right)\|_{0,\Gamma_C^i}}{\|\gamma^{\frac{1}{2}}\sigma_n^i(\mathbf{u}_{ref}^{hi})\|_{0,\Gamma_C^i}}, \text{ where } \mathbf{u}_{ref}^{hi} \text{ is the reference solution on } \Omega^i.$$

On figures 4, 5 and 6 the curves of relative error in percentage for Lagrange P_1 finite elements are plotted. The convergence rate in a H^1-norm is about 1 for the three values of θ which is in this case optimal, according to Theorem 2.5. On figures 7, 8 and 9 the same experiments are reported for Lagrange P_2 finite elements. The convergence rate for the three cases is about 1.5 which correspond to optimality as well.



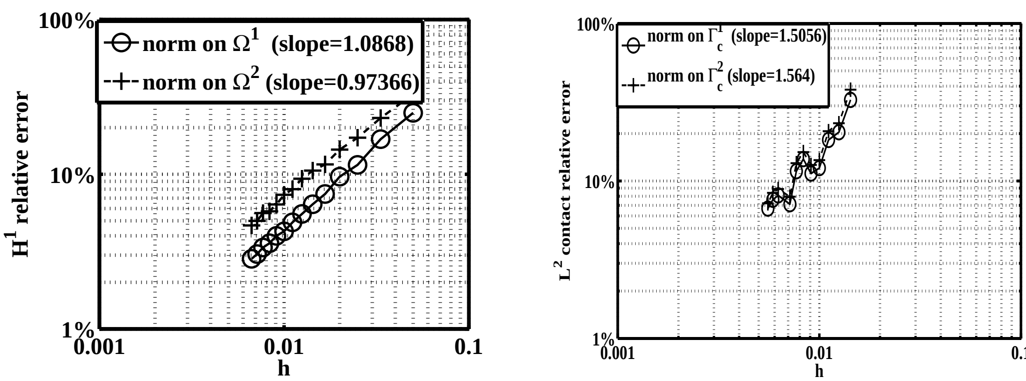
(a) (b)

Figure 4: Convergence curves in 2D for the method $\theta = 1$, with $\gamma_0 = 1/100$ and P_1 finite elements for the relative H^1-norm of the error (a) and the relative $L^2(\Gamma_C)$-norm of the error (b).



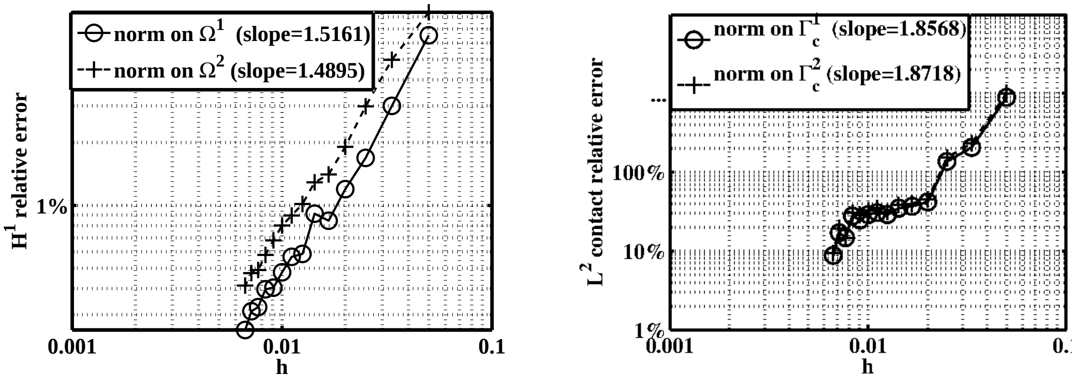
(a) (b)

Figure 5: Convergence curves in 2D for the method $\theta = 0$, with $\gamma_0 = 1/100$ and P_1 finite elements for the relative H^1-norm of the error (a) and the relative $L^2(\Gamma_C)$-norm of the error (b).



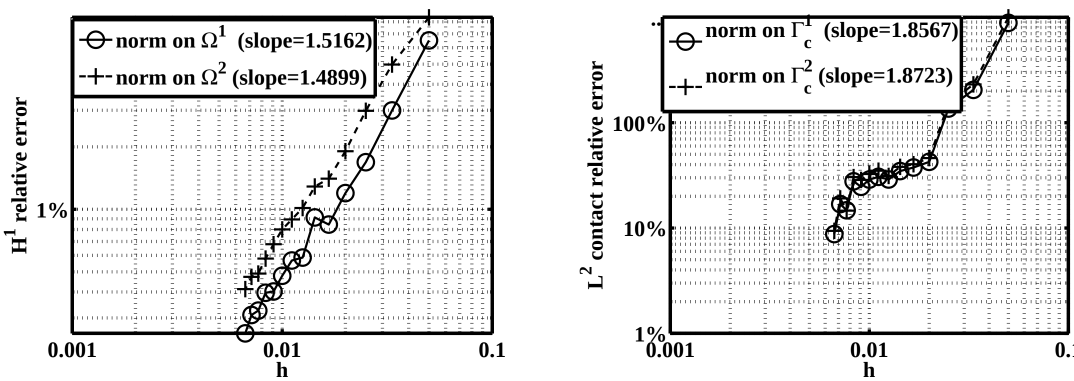
(a) (b)

Figure 6: Convergence curves in 2D for the method $\theta = -1$, with $\gamma_0 = 1/100$ and P_1 finite elements for the relative H^1-norm of the error (a) and the relative $L^2(\Gamma_C)$-norm of the error (b).



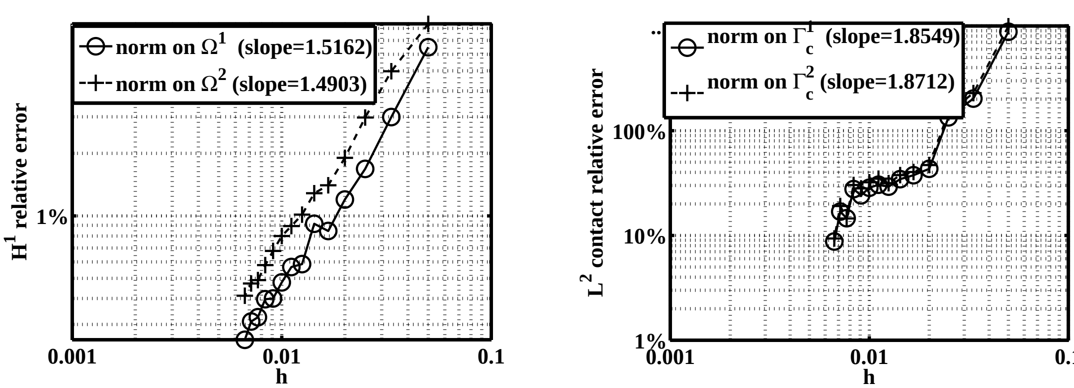
(a) (b)

Figure 7: Convergence curves in 2D for the method $\theta = 1$, with $\gamma_0 = 1/100$ and P_2 finite elements for the relative H^1-norm of the error (a) and the relative $L^2(\Gamma_C)$-norm of the error (b).



(a) (b)

Figure 8: Convergence curves in 2D for the method $\theta = 0$, with $\gamma_0 = 1/100$ and P_2 finite elements for the relative H^1-norm of the error (a) and the relative $L^2(\Gamma_C)$-norm of the error (b).

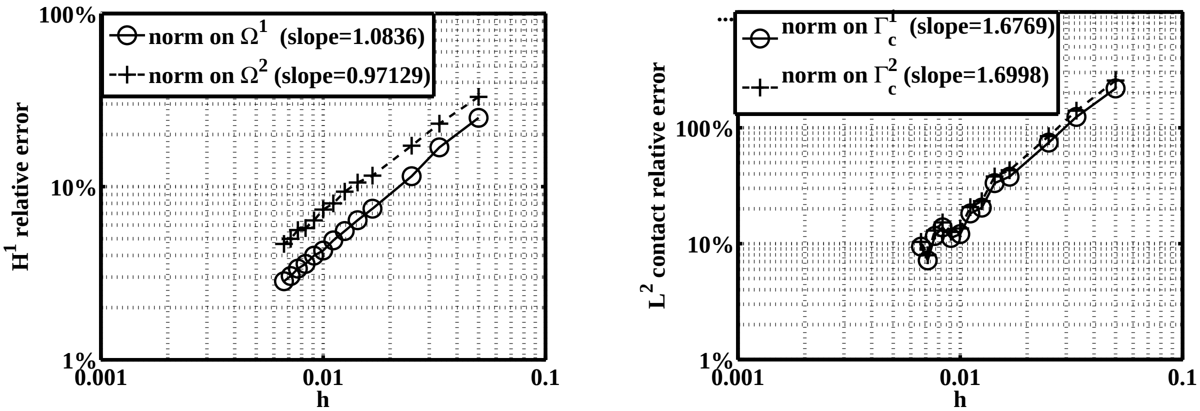


(a) (b)

Figure 9: Convergence curves in 2D for the method $\theta = -1$, with $\gamma_0 = 1/100$ and P_2 finite elements for the relative H^1-norm of the error (a) and the relative $L^2(\Gamma_C)$-norm of the error (b).

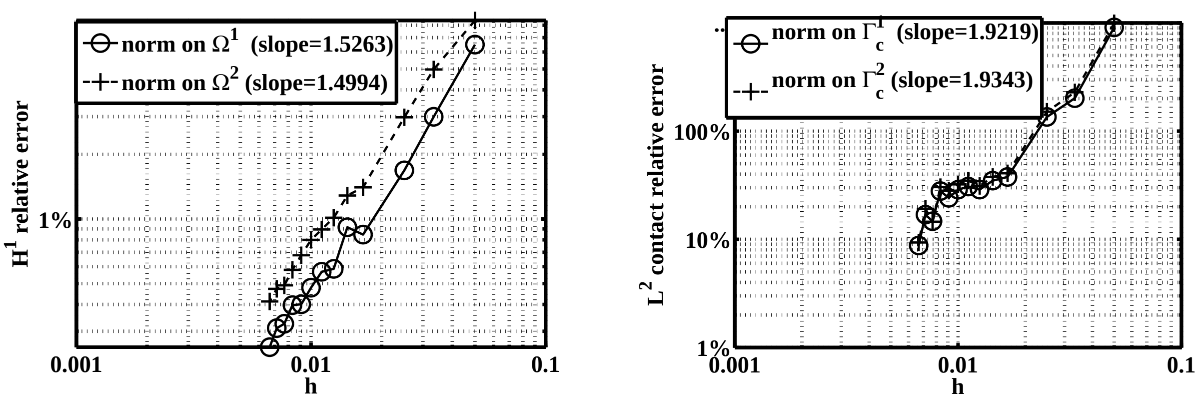
3.2 Convergence in 2D frictional contact case

We establish, as well, the convergence curves for a frictional contact (Tresca friction) with a friction coefficient $s1 = 0.1$ with the method $\theta = -1$, for a Nitsche's parameter $\gamma_0 = \frac{1}{100}$. The frictional contact curves are presented for P_1 and P_2 Lagrange elements in figures 10 and 11. Similar curves are obtained with other values of θ. The optimal convergence is obtained, as well in the frictional case with a convergence rate close to the frictionless case.



(a) (b)

Figure 10: Convergence curves in 2D frictional case for the method $\theta = -1$, with $\gamma_0 = 1/100$ with P_1 finite elements for the relative H^1-norm of the error (a) for the $L^2(\Gamma_C)$-norm of the error(b).

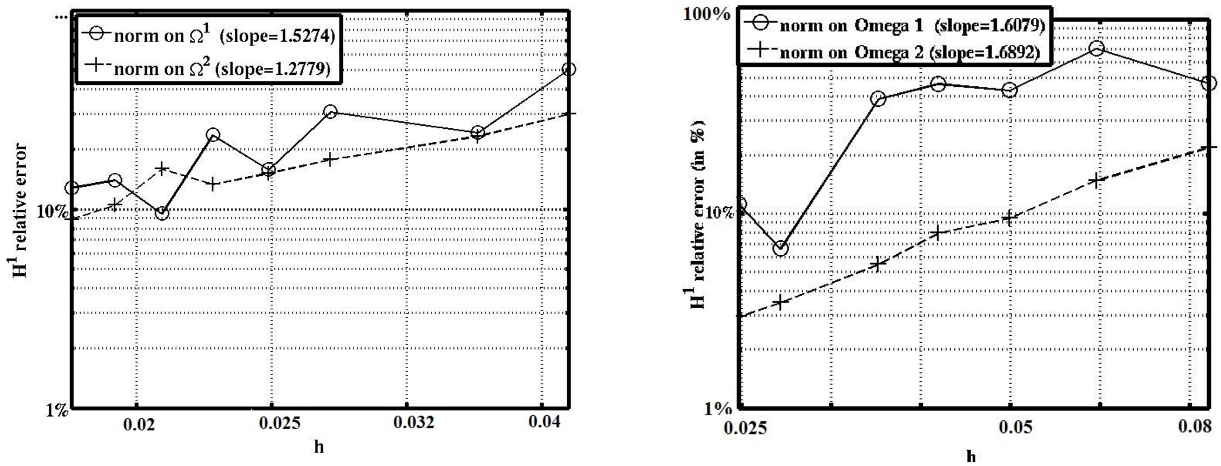


(a) (b)

Figure 11: Convergence curves in 2D frictional case for the method $\theta = -1$, with $\gamma_0 = 1/100$ with P_2 finite elements for the relative H^1-norm of the error (a) for the $L^2(\Gamma_C)$-norm of the error(b).

3.3 Convergence in the three dimensional case

The three-dimensional tests are similar to the two-dimensional ones. The error curves with $\theta = -1$ and P_1 Lagrange elements are presented in Fig. 12. Very similar conclusions can be drawn compared with the two-dimensional case.



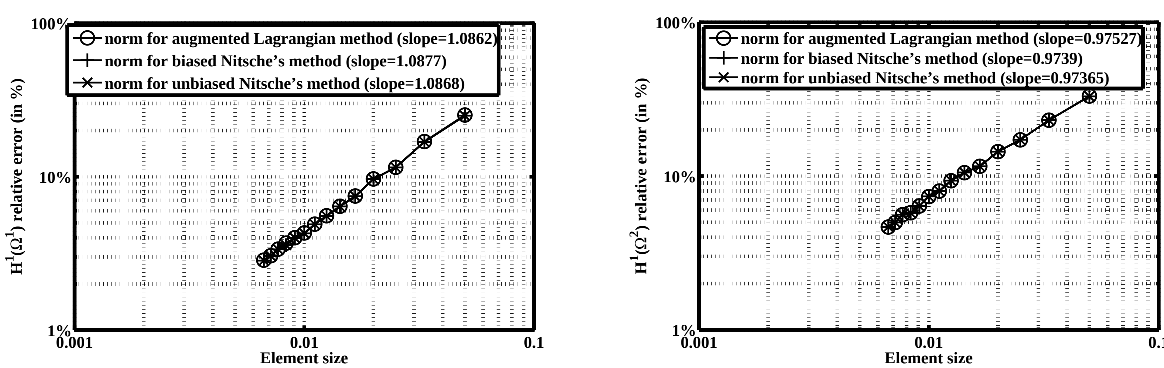
(a) (b)

Figure 12: Convergence curves in 3D for the method $\theta = -1$, with $\gamma_0 = 1/100$ for the relative H^1-norm of the error with P_1 finite elements (a) and P_2 finite elements(b) .

As expected the optimal convergence is obtained in H^1 and $L^2(\Gamma_C)$-norm for all methods in good accordance with Theorem 2.5.

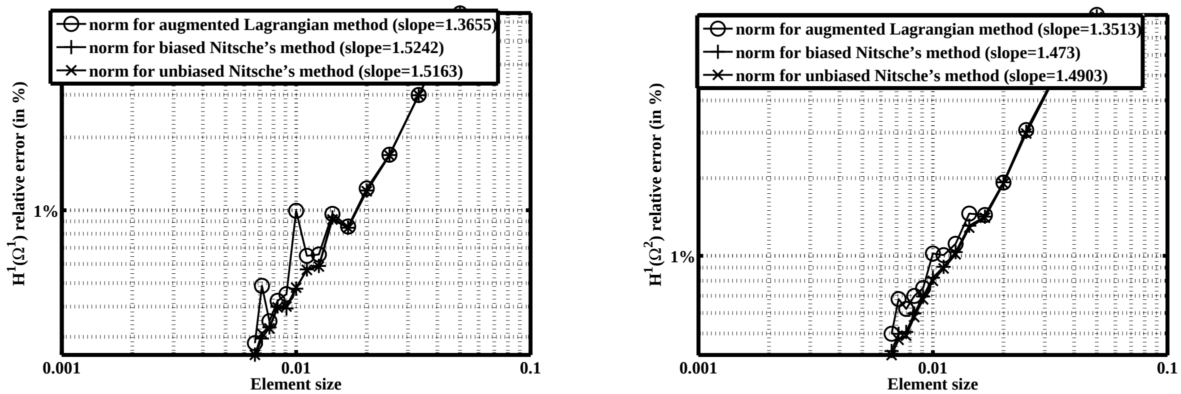
3.4 Comparison with other methods

To better compare the proposed method with other methods we present in the following the convergence curves of our test case with the convergence curves of the biased Nitsche's formulation and the augmented Lagrangian method ([7, 15]).
The curves are exactly the same for P_1 elements and very similar for P_2 ones and the convergence rate of the unbiased Nitsche's method is equal to other formulations' rate. We note that, for different values of θ the convergence is obtained for Nitsche's method (biased and unbiased) and the augmented Lagrangian method generally with a close number of iterations of the Newton algorithm.



(a) (b)

Figure 13: Comparison of convergence curves in 2D frictionless case for the method $\theta = -1$, with $\gamma_0 = 1/100$ and P_1 finite elements for the relative H^1-norm of the error on Ω^1 (a) and on Ω^2 (b) for different formulations of contact.

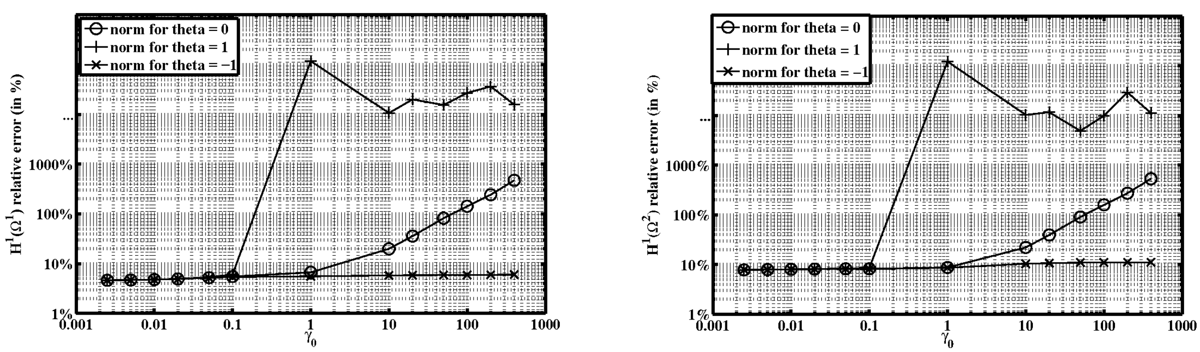


(a) (b)

Figure 14: Comparison of convergence curves in 2D frictionless case for the method $\theta = -1$, with $\gamma_0 = 1/100$ and P_2 finite elements for the relative H^1-norm of the error on Ω^1 (a) and on Ω^2 (b) for different formulations of contact.

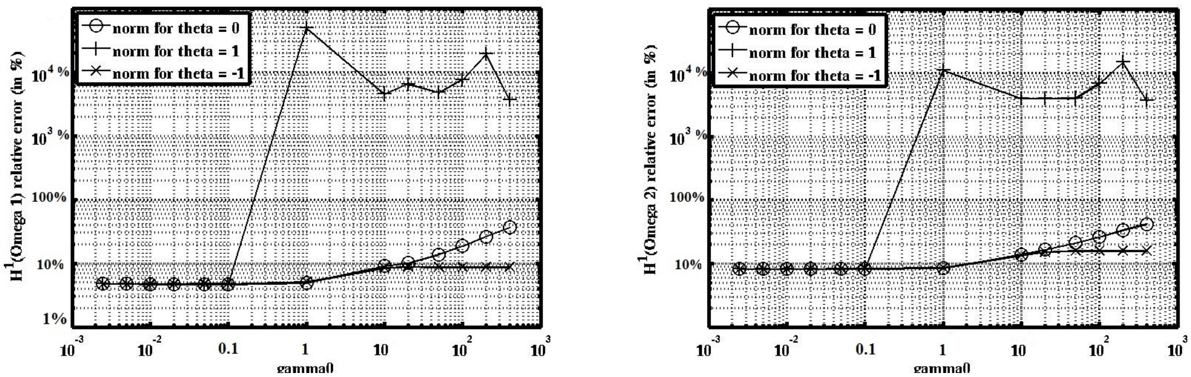
3.5 Influence of the parameter γ_0

The influence of γ_0 on the H^1-norm of the error for P_2 elements is plotted in Figure 15 in the frictionless case and on Figure 16 with a friction coefficient $s^1 = 0.1$. It is remarkable that the error curves for the smallest value of γ_0 are rather the same for the three values of θ.
The variant $\theta = 1$ is the most influenced by the value of γ_0. It converges only for γ_0 very small ($\leq 10^{-1}$). The method for $\theta = 0$ gives a much large window of choice of γ_0 though it has to remain small to keep a good solution. In agreement with the theoretical result of Theorem 2.5, the influence of γ_0 on the method $\theta = -1$ is limited.



(a) (b)

Figure 15: Influence of γ_0 on the H^1-norm error for different values of θ in the 2D frictionless case and with P_2 finite elements on Ω^1 (a) and on Ω^2 (b).



(a) (b)

Figure 16: Influence of γ_0 on the H^1-norm error for different values of θ in the 2D frictional case and with P_2 finite elements on Ω^1 (a) and on Ω^2 (b).

A strategy to guarantee an optimal convergence is of course to consider a sufficiently small γ_0. However, the price to pay is an ill-conditioned discrete problem. The study presented in [21] shows that Newton's method has important difficulties to converge when γ_0 is very small because the nonlinear discrete system (20) becomes very stiff in this case.

The accuracy of the method was discussed for the different cases with respect to the finite element used, the mesh size, the value of the parameters θ and γ_0 and the friction coefficient in the two and three dimension cases. The theoretical results are, generally, confirmed by numerical tests, especially the optimal convergence and the influence of the parameter γ_0.

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